Quantum Spin-1/2 Antiferromagnetic Chains and Strongly Coupled Multiflavor Schwinger Models

F. Berruto[(a)], **G. Grignani**[(a)], **G. W. Semenoff**[(b)] **and P. Sodano**[(a)] [1]

(a)*Dipartimento di Fisica and Sezione I.N.F.N., Università di Perugia, Via A. Pascoli I-06123 Perugia, Italy*

(b) *Department of Physics and Astronomy*
University of British Columbia
6224 Agricultural Road
Vancouver, British Columbia, Canada V6T 1Z1

Abstract

We review the correspondence between strongly coupled lattice multiflavor Schwinger models and $SU(\mathcal{N})$ antiferromagnetic chains. We show that finding the low lying states of the gauge models is equivalent to solving an $SU(\mathcal{N})$ Heisenberg antiferromagnetic chain. For the two-flavor lattice Schwinger model the massless excitations correspond to gapless states of the Heisenberg chain, while the massive states are created by fermion transport in the ground state of the spin chain. Our analysis shows explicitly how spinons may arise in lattice gauge theories.

1 Introduction

Though it is far from the scaling regime, the strong coupling limit is often used to study the qualitative properties of gauge field theories [1]. Two important features of the spectrum of non-Abelian gauge theories appear there. The strong coupling limit exhibits confinement in a rather natural way. Furthermore with certain numbers of flavors and colors of dynamical quarks, it is straightforward to show that they also exhibit chiral symmetry breaking. This has motivated several more quantitative investigations of gauge theories using strong coupling techniques and there have been attempts to compute the mass spectrum of realistic models such as quantum chromodynamics (QCD) [2, 3].

It has been recognized for some time that the strong coupling limit of lattice gauge theory with dynamical fermions is related to certain quantum spin systems. This is particularly true in the hamiltonian formalism and had already appeared in some of the earliest analyses of chiral symmetry breaking in the strong coupling limit [4]. On the other side, it has been noted that there are several similarities between some condensed matter systems with lattice fermions and lattice gauge theory systems, particularly in their strong coupling limit. For example, it is well known that the quantum spin-1/2 Heisenberg antiferromagnet is equivalent to the strong coupling limit of either $U(1)$ or $SU(2)$ lattice gauge theory [5, 6, 7, 8, 9].

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Moreover, the staggered fermions which are used to put the Dirac operator on a lattice resemble ordinary lattice fermions used in tight binding models in condensed matter physics when the latter have an half-filled band and are placed in a background $U(1)$ magnetic field π $(mod\ 2\pi)$ (1/2 of a flux quantum) through every plaquette of the lattice. This is an old result for $d = 2+1$ [10] where it was already recognized in the first work on the Azbel-Wannier-Hofstaeder problem and has since been discussed in the context of the so-called flux phases of the Hubbard model. It is actually true for all $d \geq 2+1$ [11], *i.e.* staggered fermions with the minimal number of flavors are identical to a simple nearest neighbor hopping problem with a background magnetic field which has 1/2 flux quanta per plaquette. In the condensed matter context, the magnetic flux can be produced by a condensate, as in the flux phases. As an external magnetic field, for ordinary lattice spacing it is as yet an experimentally inaccessible flux density. However, it could be achieved in analog experiments where macroscopic arrays of Josephson junctions, for example, take the place of atoms at lattice sites.

In [12] it was shown that the strong coupling limit of d-dimensional QED with $2^d/2^{[d/2]}$ flavors of fermions can be mapped onto the $S = 1/2$ quantum Heisenberg antiferromagnet in $d-1$ dimensions. The chiral symmetry breaking phase transition corresponds to a transition between the flux phase and the conventional Néel ordered phase of an antiferromagnet.

In this lectures we shall discuss the correspondence between lattice gauge theories and spin systems by analyzing, as a concrete example, the mapping of the strongly coupled multiflavor lattice Schwinger models $-$ *i.e.* QED_2 with massless fermions [13] $-$ into the antiferromagnetic quantum spin-1/2 Heisenberg chains. In the one-flavor Schwinger model, the anomaly is realized in the lattice theory via spontaneous breaking of a residual chiral symmetry [14]. In the multiflavor models, at variance with the one-flavor case, the chiral symmetry is explicitly broken by the staggered fermions, and the non-zero vacuum expectation value of a pertinent condensate is the only relic on the lattice of the chiral anomaly in the continuum [15, 16].

In section 2 we introduce the antiferromagnetic quantum spin-1/2 chains; we review the Bethe ansatz [17] solution of the Heisenberg chain [18] and compare it with the exact diagonalization of 4, 6 and 8 sites chains. Furthermore we generalize the analysis to the spin-1/2 $SU(\mathcal{N})$ antiferromagnetic Heisenberg chains [19, 15, 16].

Section 3 is devoted to a detailed investigation of the two-flavor lattice Schwinger model in the strong coupling limit using the hamiltonian formalism and staggered fermions. We show that the problem of finding the low lying states is equivalent to solving the Heisenberg antiferromagnetic spin chain. The massless excitations of the gauge model correspond to gapless states of the Heisenberg chain, while the massive excitations are created by fermion transport in the ground state of the spin chain and we compute their masses in terms of vacuum expectation values (VEV's) of spin-spin correlation functions [19, 15, 16, 14].

In section 4 we study the generic $\mathcal{N}$-flavor lattice Schwinger models and their mapping into spin-1/2 quantum $SU(\mathcal{N})$ antiferromagnetic spin chains. The analysis strictly parallels what we found in section 3, but some rather surprising difference arises depending on if $\mathcal{N}$ is even or odd.

2 Antiferromagnetic spin chains

The title of this section deserves an entire book to be properly developed, and in fact some books, reviews and long articles about spin chains do already exist [20]. Spin chains interested mathematical physicists for their exact solvability, field theorists for the possibility to test their methods by comparation with exact results and condensed matter physicists for the possibility to understand the spectra of real physical systems. It is our purpose to report a self-contained analysis of antiferromagnetic spin chains, useful to understand section 3 and 4 where we shall show that the multiflavor lattice Schwinger models in the strong coupling limit are effectively described by the spin-1/2 $SU(\mathcal{N})$ antiferromagnetic Heisenberg chains.

A common tool to study spin systems is spin-wave theory. Spin-wave theory was developed by Anderson, Kubo and others [21] in the 1950s and it is a very powerful approach to quantum

magnetic systems in dimensions greater than one, predicting long-range order and gapless Goldstone bosons. The situation remained clouded for magnetic chains because it was known that long-range order could not occur in one dimension [22], but the Bethe ansatz predicts massless excitations. There is "quasi-long-range-order" corresponding to a power law decay of spin-spin correlators and there are gapless excitations which are not true Goldstone bosons.

Quantum spin chains have been extensively studied in the literature, starting from the seminal paper by Hans Bethe [17] in 1931 for the spin-1/2 case, where he introduced an ansatz to write down the eigenfunctions of the Heisenberg Hamiltonian, describing a chain with periodic boundary conditions. Seven years later Hulthen [23] was able to compute the ground state energy of the antiferromagnetic Heisenberg chain. We had to wait until 1984 to know exactly the spectrum of this model, when Faddeev and Takhtadzhyan [18] analyzed the model using the algebraic Bethe ansatz and showed that the only one-particle excitation is a doublet of spin-1/2 quantum excitations with gapless dispersion relation. This excitation is a kink rather than an ordinary particle and is called spinon. All the eigenstates of the antiferromagnetic Heisenberg Hamiltonian contain an even number of kinks, nevertheless the kinks are localizable objects and one can consider their scattering. There are no bound states of kinks in this model.

The one dimensional spin systems are not only unusual because they are exactly solvable but also for their incompatibility with long range order. Actually solvability and absence of long range order are deeply related concepts; systems with spontaneously broken symmetries are more difficult to describe and resist analytical solutions. All isotropic antiferromagnetic quantum spin chains with short range interactions exhibit quantum spin-liquid ground states $-$ *i.e.* states with short range antiferromagnetic correlations and no order. Moreover these chains have strange quantum elementary excitations above these ground states, that are not ordinary spin waves but are usually called spinons $-$ neutral spin-1/2 kinks.

As R. B. Laughlin [24] points out: "spinons are not "like" anything familiar to most of us, but are instead an important and beautiful instance of fractional quantization, the physical phenomenon in which particles carrying pieces of a fundamental quantum number, such as charge or spin, are created as a collective motion of many conventional particles obeying quantum mechanical laws. The fractional quantum number of the spinon is its spin. It is fractional because the particles out of which the magnetic states are constructed are spin flips, which carry spin 1."

One significant result of our analysis [19, 15, 16] has been to show explicitly how spinons appear in lattice gauge theories. More precisely we showed that the massless excitations of the two-flavor Schwinger model coincide with the spinons of the spin-1/2 antiferromagnetic Heisenberg model.

Section (2.1) is devoted to review the Bethe ansatz solution of the antiferromagnetic Heisenberg chain. We shall compare the exact solution given in [18] with a study of finite size chains of 4,6, and 8 sites in section (2.2). An original result presented is the thermodynamic limit coefficient of the states with $N-2$ domain walls composing the ground state.

The spin-spin correlation function $\sum_x < g.s.|\vec{S}_x \cdot \vec{S}_{x+2}|g.s. >$ originally computed by M. Takahashi [25], is discussed in section (2.3). The vacuum expectation value of the square of the vector $\vec{V} = \sum_x \vec{S}_x \wedge \vec{S}_{x+1}$ is also computed.

In section (2.4) we introduce the spin-1/2 $SU(\mathcal{N})$ antiferromagnetic Heisenberg chains. To provide an intuitive picture we study the $SU(3)$ two sites chain and we find the ground state. A short review of the literature on $SU(\mathcal{N})$ antiferromagnetic Heisenberg chains is provided.

2.1 The Bethe Ansatz solution of the antiferromagnetic Heisenberg chain

In this section the Bethe ansatz solution of the antiferromagnetic Heisenberg chain [18] is discussed in detail. Moreover we shall compare the exact solution given by L.D. Faddeev and L.A. Takhtadzhyan in [18] with a study of finite size chains of 4, 6 and 8 sites. We show that already these very small finite systems exhibit spectra that match very well with the thermodynamic limit solution. We suggest to the reader interested in the subject the references [20].

The Bethe ansatz is a method of solution of a number of quantum field theory and statistical mechanics models in two space-time dimensions. This method was first suggested by Bethe [17] in 1931 from which takes its name. Historically one can call this formulation the Coordinate Bethe ansatz to distinguish it from the modern formulation known as Algebraic Bethe ansatz. The eigenfunctions of some (1+1)-dimensional Hamiltonians can be constructed imposing periodic boundary conditions which lead to a system of equations for the permitted values of momenta. These are known as the Bethe equations which are also useful in the thermodynamic limit. The energy of the ground state may be calculated in this limit and its excitations can be investigated.

We review here the method applyied to the study of the antiferromagnetic Heisenberg chain [18]. In particular it is shown that there is only one excitation with spin-1/2 which is a kink: physical states have only an even number of kinks, therefore they always have an integer spin. The one dimensional isotropic Heisenberg model describes a system of N interacting spin-$\frac{1}{2}$ particles. The Hamiltonian of the model is

$$H_J = J \sum_{x=1}^{N} (\vec{S}_x \cdot \vec{S}_{x+1} - \frac{1}{4}) \quad . \tag{2.1}$$

where $J > 0$ ($J < 0$ would describe a ferromagnet) and the spin operators have the following form

$$\vec{S}_x = 1_1 \otimes 1_2 \otimes \ldots \otimes \frac{\vec{\sigma}_x}{2} \otimes \ldots \otimes 1_N \quad . \tag{2.2}$$

They act nontrivially only on the Hilbert space of the x^{th} site. Periodic boundary conditions are assumed.

The Hamiltonian (2.1) is invariant under global rotations in the spin space, generated by

$$\vec{S} = \sum_{x=1}^{N} \vec{S}_x \quad . \tag{2.3}$$

Due to the periodic boundary conditions, under translations generated by the operator $\hat{T}$,

$$\hat{T} \quad : \vec{S}_x \longrightarrow \vec{S}_{x-1} \tag{2.4}$$

the Hamiltonian is invariant and $[\vec{S}, \hat{T}] = 0$.

In order to diagonalize H_J it is convenient to use an eigenfunction basis of operators commuting with H_J, so obviously $\vec{S}^2$, S^z and also $\hat{T}$. Let us sketch the Coordinate Bethe ansatz technique. One has to introduce the "false vacuum"

$$|\Omega> = \prod_{x=1}^{N} \otimes | \uparrow>_x \tag{2.5}$$

with

$$S_x^+ | \uparrow>_x \quad = \quad 0 \tag{2.6}$$

$$S_x^3 | \uparrow>_x \quad = \quad \frac{1}{2} | \uparrow>_x \tag{2.7}$$

where

$$S_x^{\pm} = S_x^1 \pm S_x^2 \tag{2.8}$$

$$S^3 |\Omega> \quad = \quad \frac{N}{2} |\Omega> \tag{2.9}$$

$$\vec{S}^2 |\Omega> \quad = \quad \frac{N}{2} (\frac{N}{2} + 1) |\Omega> \tag{2.10}$$

$$\hat{T} |\Omega> \quad = \quad |\Omega> \quad . \tag{2.11}$$

All the other basis vectors have $S^3 < \frac{N}{2}$ and one can get them by properly acting on $|\Omega>$ with the lowering operators S^-_x. Let us start from the generic state with $S^3 = \frac{N}{2} - 1$, with $N-1$ spins up and $M = 1$ spin down

$$|M=1>=\sum_{x=1}^{N}\phi_x|x> \quad with \quad |x>=S^-_x|\Omega> \tag{2.12}$$

where the coefficients ϕ_x must be such that $|M=1>$ is a translationally and rotationally invariant state

$$\hat{T}|M=1>=\sum_{x=1}^{N}\phi_x|x-1>=\sum_{x=1}^{N}\phi_{x+1}|x>=\mu\sum_{x=1}^{N}\phi_x|x> \tag{2.13}$$

and from Eq.(2.13) one gets

$$\phi_{x+1}=\mu\phi_x \tag{2.14}$$

$$\begin{aligned}\phi_{x+1} &= \mu^x\phi_1 \qquad x\neq N & (2.15)\\ \phi_1 &= \mu\phi_N=\mu^N\phi_1\longrightarrow\mu^N=1 \quad . & (2.16)\end{aligned}$$

Setting $\phi_1 = 1$ one has $\phi_x = \mu^{x-1}$. There are N possible values for ϕ_x, corresponding to the N roots of the unity. One of these roots corresponds to a state with $S = \frac{N}{2}$

$$\sum_{x=1}^{N}|x>=S^-|\Omega> \quad\longrightarrow\quad \mu=1 \tag{2.17}$$

while the other $N-1$ roots have $S = \frac{N}{2} - 1$.

The generic case with M spins down is more complicated. Let us consider the case $M = 2$ to understand what happens

$$|M=2>=\sum_{x_1<x_2=1}^{N}\phi(x_1,x_2)|x_1,x_2> \quad with \quad |x_1,x_2>=S^-_{x_1}S^-_{x_2}|\Omega> \quad . \tag{2.18}$$

By requiring the translational invariance of the state $|M = 2>$ one has

$$\hat{T}|M=2>=T|M=2> \tag{2.19}$$

and for $x_2 < N$ one has

$$\phi(x_1+1,x_2+1)=T\phi(x_1,x_2) \tag{2.20}$$

that would easily give

$$\phi(x_1,x_2)=\mu_1^{x_1-1}\mu_2^{x_2-1} \quad , \quad T=\mu_1\mu_2 \tag{2.21}$$

but due to the periodic boundary conditions one has to find coefficients $\phi(x_1, x_2)$ that satisfy not only Eq.(2.20) but also

$$\phi(1,x+1)=T\phi(x,N) \quad . \tag{2.22}$$

Eq.(2.22) is no more satisfied by (2.21). Bethe proposed the following ansatz for the coefficients $\phi(x_1, x_2)$

$$\phi(x_1,x_2)=A_{1,2}\mu_1^{x_1-1}\mu_2^{x_2-1}+A_{2,1}\mu_2^{x_1-1}\mu_1^{x_2-1} \quad . \tag{2.23}$$

Eq.(2.20,2.22) are satisfied if the following equations hold

$$\begin{aligned}A_{12} &= A_{21}\mu_1^N & (2.24)\\ A_{21} &= A_{12}\mu_2^N \quad . & (2.25)\end{aligned}$$

By imposing to $|M = 2>$ to be an highest weight state

$$S^+|M=2>=0 \tag{2.26}$$

taking into account Eq.(2.24,2.25) and introducing the following change of variables

$$\mu_\alpha = \frac{\lambda_\alpha - \frac{i}{2}}{\lambda_\alpha - \frac{i}{2}} \tag{2.27}$$

one gets the so called Bethe ansatz equations

$$(\frac{\lambda_\alpha - \frac{i}{2}}{\lambda_\alpha + \frac{i}{2}})^N = -\prod_{\beta=1}^{2} \frac{\lambda_\alpha - \lambda_\beta - i}{\lambda_\alpha - \lambda_\beta + i} \quad . \tag{2.28}$$

In the general case of M spins flipped one gets

$$(\frac{\lambda_\alpha - \frac{i}{2}}{\lambda_\alpha + \frac{i}{2}})^N = -\prod_{\beta=1}^{M} \frac{\lambda_\alpha - \lambda_\beta - i}{\lambda_\alpha - \lambda_\beta + i} \quad . \tag{2.29}$$

The energy and the momentum of a given state with M spins down can be expressed in terms of the parameters λ_α

$$E_M = \sum_{\alpha=1}^{M} \epsilon_\alpha = -\frac{J}{2}\sum_{\alpha=1}^{M} \frac{1}{\lambda_\alpha^2 + \frac{1}{4}} \tag{2.30}$$

$$P_M = i \ln T = \sum_{\alpha=1}^{M} p_\alpha = i \sum_{\alpha=1}^{M} \ln \frac{\lambda_\alpha - \frac{i}{2}}{\lambda_\alpha + \frac{i}{2}} \quad . \tag{2.31}$$

Energy and momentum are thus additive as if there were M independent particles and the λ_α must satisfy the Bethe ansatz equations (2.29) in order for E_M and P_M to be eigenvalues of the Hamiltonian and momentum operators.

The solution of the antiferromagnetic Heisenberg chain is reduced to the solution of the system of the M algebraic equations (2.29). This, in general, is not an easy task. It can be shown [18], however, that, in the thermodynamic limit $N \to \infty$, the complex parameters λ have the form

$$\lambda_l = \lambda_{j,L} + il \quad , \quad l = -L, -L+1, \ldots, L-1, L; \tag{2.32}$$

where L is a non-negative integer or half-integer, $\lambda_{j,L}$ is the real part of the solution of (2.29) and we shall define shortly the set of allowed values for the integer index j. The λ's that, for a given $\lambda_{j,L}$, are obtained varying l between $[-L, L]$ by integer steps, form a string of length $2L+1$, see fig.(1). This arrangement of λ's in the complex plane is called the "string hypothesis" [18]. In the following we shall verify that, even on finite size systems, the "string hypothesis" is very well fulfilled.

In a generic Bethe state with M spins down, there are M solutions to (2.29), which can be grouped according to the length of their strings. Let us denote by ν_L the number of strings of length $2L+1$, $L = 0, \frac{1}{2}, \ldots$; strings of the same length are obtained by changing the real parts, $\lambda_{j,L}$, of the λ's in (2.32); as a consequence $j = 1, \ldots, \nu_L$. If one denotes the total number of strings by q one has

$$q = \sum_L \nu_L \quad , \quad M = \sum_L (2L+1)\nu_L \quad . \tag{2.33}$$

The set of integers $(M, q, \{\nu_L\})$ constrained by (2.33), characterizes Bethe states up to the fixing of the q numbers $\lambda_{j,L}$; this set is called the "configuration". Varying M, q and ν_L, one is able to construct all the 2^N eigenstates of an Heisenberg antiferromagnetic chain of N sites [18]. The energy and momentum of the Bethe's state, corresponding to a given configuration $-$ within exponential accuracy as $N \to \infty$ $-$ consist of q summands representing the energy and momentum of separate strings. For the parameters $\lambda_{j,L}$ of the given configuration, taking the logarithm of (2.29) the following system of equations is obtained in the thermodynamic limit

$$2N \arctan \frac{\lambda_{j,L_1}}{L_1 + \frac{1}{2}} = 2\pi Q_{j,L_1} + \sum_{L_2} \sum_{k=1}^{\nu_{L_2}} \Phi_{L_1 L_2}(\lambda_{j,L_1} - \lambda_{k,L_2}) \quad , \tag{2.34}$$

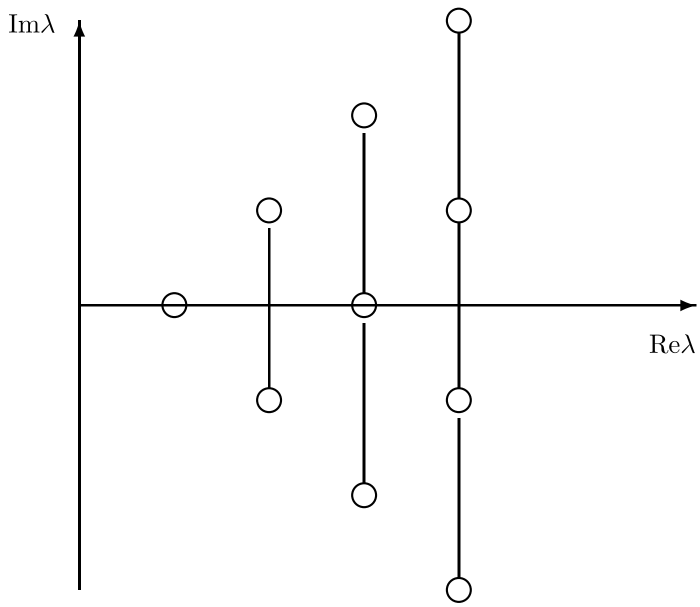

Figure 1: Strings for $L = 0, \frac{1}{2}, 1, \frac{3}{2}$

where

$$\Phi_{L_1 L_2}(\lambda) = 2 \sum_{L=|L_1-L_2|\neq 0}^{L_1+L_2} (\arctan\frac{\lambda}{L} + \arctan\frac{\lambda}{L+1}) \quad . \tag{2.35}$$

Integer and half integer numbers $Q_{j,L}$ parametrize the branches of the arcotangents and, consequently, the possible solutions of the system of Eqs.(2.34). In ref.[18] it was shown that the $Q_{j,L}$ are limited as

$$-Q_L^{max} \leq Q_{1,L} < Q_{2,L} < \ldots < Q_{\nu_L,L} \leq Q_L^{max} \tag{2.36}$$

with Q_L^{max} given by

$$Q_L^{max} = \frac{N}{2} - \sum_{L'} J(L,L')\nu_{L'} - \frac{1}{2} \tag{2.37}$$

and

$$J(L_1,L_2) = \begin{cases} 2\min(L_1,L_2)+1 & \text{if } L_1 \neq L_2 \\ 2L_1 + \frac{1}{2} & \text{if } L_1 = L_2 \quad . \end{cases} \tag{2.38}$$

The admissible values for the numbers $Q_{j,L}$ are called the "vacancies" and their number for every L is denoted by P_L

$$P_L = 2Q_L^{max} + 1 \quad . \tag{2.39}$$

The main hypothesis formulated in [18] is that to every admissible collection of $Q_{j,L}$ there corresponds a unique solution of the system of equations (2.34). The solution always provides, in a multiplet, the state with the highest value of the third spin component S^3.

Let us now consider some simple example. The simplest configuration has only strings of length 1, *i.e.* all the λ's are real. The singlet associated to this configuration

$$M = q = \nu_0 = \frac{N}{2} \quad , \quad \nu_L = 0 \quad , \quad L > 0 \quad , \tag{2.40}$$

is the ground state. The vacancies of the strings of length 1, *i.e.* the admissible values of $Q_{j,0}$, due to eqs.(2.36,2.37,2.38), belong to the segment

$$-\frac{N}{4} + \frac{1}{2} \leq Q_{j,0} \leq \frac{N}{4} - \frac{1}{2} \quad . \tag{2.41}$$

Therefore they are $N/2$. All these vacancies must then be used to find the $N/2$ strings of length 1. As a consequence this state is uniquely specified and no degeneracy is possible.

Next we consider the configuration that provides a singlet with 1 string of length 2 and all the others of length 1:

$$M = \frac{N}{2} \quad , \quad q = \frac{N}{2} - 1 \quad , \quad \nu_0 = \frac{N}{2} - 2 \quad , \quad \nu_{\frac{1}{2}} = 1 \quad , \quad \nu_L = 0 \quad , \quad L > \frac{1}{2} \ . \tag{2.42}$$

For the strings of length 1 the number of vacancies is again $N/2$; for the string of length 2 there is one vacancy and the only admissible $Q_{j,1}$ equals 0. Thus, since the number of strings of length 1 is $\nu_0 = \frac{N}{2} - 2$, there are two vacancies for which Eqs.(2.34) have no solution; they are called "holes" and are denoted $Q_1^{(h)}$ and $Q_2^{(h)}$. This configuration is determined by two parameters: the positions of two "holes" which vary independently in the interval (2.41).

There is another state with only 2 holes: the triplet corresponding to the configuration

$$M = q = \nu_0 = \frac{N}{2} - 1 \quad , \quad \nu_L = 0 \quad , \quad L > 0 \tag{2.43}$$

The number of vacancies for the strings of length 1 equals $\frac{N}{2} + 1$, while $\nu_0 = \frac{N}{2} - 1$.

The excitations determined by the configurations (2.42,2.43) belong to the configuration class called in [18] $\mathcal{M}_{\mathcal{AF}}$. The class $\mathcal{M}_{\mathcal{AF}}$ is characterized as follows: the number of strings of length 1 in each configuration belonging to this class differs by a finite quantity from $N/2$, $\nu_0 = \frac{N}{2} - k_0$ where k_0 is a positive finite constant, so that the number of strings of length greater than 1 is finite. From (2.39) one then has

$$P_0 \;=\; \frac{N}{2} + k_0 - 2\sum_{L>0} \nu_L \tag{2.44}$$

$$P_L \;=\; 2k_0 - 2\sum_{L'>0} J(L,L')\nu_{L'} \ , \quad L > 0 \tag{2.45}$$

so that

$$P_0 \geq \frac{N}{2} \ , \quad P_L < 2k_0 \ , \quad L > 0 \ . \tag{2.46}$$

From (2.44) follows that the number of holes for the strings of length 1 is always even and equals 2 only for the singlet and the triplet excitations discussed above. One can imagine the class $\mathcal{M}_{\mathcal{AF}}$ as a "sea" of strings of length 1 with a finite number of strings of length greater than 1 immersed into it. It was proven in [18] that, in the thermodynamic limit, the states belonging to $\mathcal{M}_{\mathcal{AF}}$ have finite energy and momentum with respect to the antiferromagnetic vacuum, whereas each of the states which corresponds to a configuration not included in the class $\mathcal{M}_{\mathcal{AF}}$ has an infinite energy relative to the antiferromagnetic vacuum.

Let us now sketch the computation of the thermodynamic limit ground state energy. Eqs.(2.34) for the ground state have the form

$$\arctan 2\lambda_j = \frac{\pi Q_j}{N} + \frac{1}{N}\sum_{k=1}^{N/2} \arctan(\lambda_j - \lambda_k) \quad . \tag{2.47}$$

Taking the thermodynamic limit $N \to \infty$, one has

$$\frac{Q_j}{N} \to x \quad , \quad -\frac{1}{4} \leq x \leq \frac{1}{4} \quad , \quad \lambda_j \to \lambda(x) \quad , \tag{2.48}$$

and Eqs.(2.47) can be rewritten in the form

$$\arctan 2\lambda(x) = \pi x + \int_{-\frac{1}{4}}^{\frac{1}{4}} \arctan(\lambda(x) - \lambda(y))dy \quad . \tag{2.49}$$

Upon introducing the density of the numbers $\lambda(x)$ in the interval $d\lambda$

$$\rho(\lambda) = \frac{1}{\frac{d\lambda(x)}{dx}|_{x=x(\lambda)}} \tag{2.50}$$

and differentiating Eqs.(2.49), one gets

$$\rho(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}|\xi|}}{1+e^{-|\xi|}} e^{-i\lambda|\xi|} d\xi = \frac{1}{2\cosh \pi\lambda} \quad . \tag{2.51}$$

The density $\rho(\lambda)$ introduced in this way is normalized to 1/2. It is now easy to compute the energy and the momentum of the ground state

$$E_{g.s.} = \sum_{\alpha=1}^{\frac{N}{2}} \epsilon_\alpha = N\int_{-\infty}^{\infty} \epsilon(\lambda)\rho(\lambda)d\lambda = -\frac{JN}{4}\int_{-\infty}^{\infty} d\lambda \frac{1}{\left(\lambda^2+\frac{1}{4}\right)\cosh\pi\lambda} = -JN\ln 2 \tag{2.52}$$

$$P_{g.s.} = \sum_{\alpha=1}^{\frac{N}{2}} p_\alpha = N\int_{-\infty}^{\infty} p(\lambda)\rho(\lambda)d\lambda = -\frac{N}{2}\int_{-\infty}^{\infty} d\lambda \frac{\pi}{\cosh\pi\lambda} = \frac{N}{2}\pi \pmod{2\pi} \quad . \tag{2.53}$$

According to Eq.(2.53), $P_{g.s.} = 0$ (mod2π) for $\frac{N}{2}$ even, and $P_{g.s.} = \pi$ (mod 2π) for $\frac{N}{2}$ odd. The ground state, as expected, is a singlet, in fact the spin S is given by

$$S = \frac{N}{2} - \sum_{\alpha=1}^{N/2} 1 = \frac{N}{2} - N\int_{-\infty}^{\infty}\rho(\lambda)d\lambda = 0 \quad . \tag{2.54}$$

Let us analyze the triplet described by Eq.(2.43); Eqs.(2.34) take the form

$$\arctan 2\lambda_j = \frac{\pi Q_j}{N} + \frac{1}{N}\sum_{k=1}^{\frac{N}{2}-1} \arctan(\lambda_j - \lambda_k) \tag{2.55}$$

where now the numbers Q_j lie in the segment $[-\frac{N}{4}, \frac{N}{4}]$ and have two holes, $Q_1^{(h)}$ and $Q_2^{(h)}$ with $Q_1^{(h)} < Q_2^{(h)}$. Taking the thermodynamic limit one gets

$$\frac{Q_1^{(h)}}{N} \to x_1 \quad , \quad \frac{Q_2^{(h)}}{N} \to x_2 \quad , \quad \frac{Q_j}{N} \to x + \frac{1}{N}(\theta(x-x_1) + \theta(x-x_2)) \tag{2.56}$$

where $\theta(x)$ is the Heaviside function. Eqs.(2.55) become

$$\arctan 2\lambda(x) = \pi x + \frac{\pi}{N}(\theta(x-x_1) + \theta(x-x_2)) + \int_{-\frac{1}{4}}^{\frac{1}{4}} \arctan(\lambda(x) - \lambda(y))dy \quad . \tag{2.57}$$

Eq.(2.57) gives, for this triplet, the density of λ, $\rho(\lambda) = \frac{d\lambda}{dx}$

$$\rho_t(\lambda) = \rho(\lambda) + \frac{1}{N}(\sigma(\lambda-\lambda_1) - \sigma(\lambda-\lambda_2)) \tag{2.58}$$

where $\rho(\lambda)$ is given in Eq.(2.51) and

$$\sigma(\lambda) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+e^{-|\xi|}} e^{-i\lambda\xi} d\xi \quad . \tag{2.59}$$

λ_1 and λ_2 are the parameters of the holes, $\lambda_i = \lambda(x_i)$, $i = 1, 2$. The energy and the momentum of this state measured from the ground state are now easily computed

$$\epsilon_T(\lambda_1, \lambda_2) = N\int_{-\infty}^{\infty} \epsilon(\lambda)(\rho_t(\lambda) - \rho(\lambda))d\lambda = \epsilon(\lambda_1) + \epsilon(\lambda_2) \tag{2.60}$$

$$p_T(\lambda_1,\lambda_2) = N\int_{-\infty}^{\infty} p(\lambda)(\rho_t(\lambda)-\rho(\lambda))d\lambda = p(\lambda_1)+p(\lambda_2) \quad (mod \quad 2\pi) \tag{2.61}$$

where

$$\epsilon(\lambda) = \int_{-\infty}^{\infty} \epsilon(\mu)\sigma(\lambda-\mu)d\mu = J\frac{\pi}{2\cosh\pi\lambda} \tag{2.62}$$

$$p(\lambda) = \int_{-\infty}^{\infty} p(\mu)\sigma(\lambda-\mu)d\mu = \arctan\sinh\pi\lambda - \frac{\pi}{2}, \quad -\pi \le p(\lambda) \le 0 \quad . \tag{2.63}$$

From Eqs.(2.62,2.63) one gets

$$\epsilon = -\frac{J\pi}{2}\sin p \quad . \tag{2.64}$$

The momentum $p_T(\lambda_1,\lambda_2)$ varies over the interval $[0,2\pi)$, when λ_1 and λ_2 run independently over the whole real axis. The spin of this state can be computed by the formula

$$S = -\int_{-\infty}^{\infty}(\sigma(\lambda-\lambda_1)+\sigma(\lambda-\lambda_2))d\lambda = 1 \quad . \tag{2.65}$$

Let us finally analize the singlet excitation characterized by the configuration (2.42). Denoting by λ_S the only number among the $\lambda_{j,1/2}$ which characterizes the string of length 2 and by λ_j the numbers $\lambda_{j,0}$ for the strings of length 1, Eqs.(2.34) read

$$\arctan 2\lambda_j = \frac{\pi Q_j}{N} + \frac{1}{N}\Phi(\lambda_j-\lambda_S) + \frac{1}{N}\sum_{k=1}^{\frac{N}{2}-2}\arctan(\lambda_j-\lambda_k) \tag{2.66}$$

$$\arctan\lambda_S = \frac{1}{N}\sum_{j=1}^{\frac{N}{2}-2}\Phi(\lambda_S-\lambda_j) \tag{2.67}$$

with

$$\Phi(\lambda) = \arctan 2\lambda + \arctan\frac{2}{3}\lambda \quad . \tag{2.68}$$

The $\frac{N}{2}-2$ numbers Q_j vary in the segment $[-\frac{N}{4}+\frac{1}{2}, \frac{N}{4}-\frac{1}{2}]$; among them there are the two holes $Q_1^{(h)}$ and $Q_2^{(h)}$. Taking the thermodynamic limit one finds the density of λ's for the singlet

$$\rho(\lambda_S) = \rho(\lambda) + \frac{1}{N}(\sigma(\lambda-\lambda_1)+\sigma(\lambda-\lambda_2)+\omega(\lambda-\lambda_S)) \tag{2.69}$$

where ρ and σ were given in Eqs.(2.51, 2.59) and where

$$\omega(\lambda) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}|\xi|-i\lambda\xi}d\xi = -\frac{2}{\pi(1+4\lambda^2)} \quad . \tag{2.70}$$

In [18] it was demonstrated that the string parameter λ_S is uniquely determined by the λ's parametrizing the two holes

$$\lambda_S = \frac{\lambda_1^{(h)}+\lambda_2^{(h)}}{2} \quad . \tag{2.71}$$

In [18] it was also proved the remarkable fact that the string of length 2 does not contribute to the energy and momentum of the excitation, so that the singlet and the triplet have the same dispersion relations

$$\epsilon_S(\lambda_1,\lambda_2) = \epsilon_T(\lambda_1,\lambda_2) = \epsilon(\lambda_1)+\epsilon(\lambda_2) \tag{2.72}$$

$$p_S(\lambda_1,\lambda_2) = p_T(\lambda_1,\lambda_2) = p(\lambda_1)+p(\lambda_2) \quad (\mathrm{mod} \quad 2\pi) \quad . \tag{2.73}$$

The spin of this excitation is, of course, zero

$$S = -2 - \int_{-\infty}^{\infty}(2\sigma(\lambda)+\omega(\lambda))d\lambda = 0 \tag{2.74}$$

The only difference between the state whose configuration is given in Eq.(2.43) and the state of Eq.(2.42) is the spin.

To summarize, the finite energy excitations of the antiferromagnetic Heisenberg chain are only those belonging to the class $\mathcal{M}_{\mathcal{AF}}$ and are described by scattering states of an even number of quasiparticles or kinks. The momentum p of these kinks runs over half the Brillouin zone $-\pi \leq p \leq 0$, the dispersion relation is $\epsilon(p) = \frac{J\pi}{2}\sin p$, Eq.(2.64), and the spin of a kink is 1/2. The singlet and the triplet excitations described above are the only states composed of two kinks, the spins of the kinks being parallel for the triplet and antiparallel for the singlet. For vanishing total momentum all the states belonging to $\mathcal{M}_{\mathcal{AF}}$ have the same energy of the ground state so that they are gapless excitations. Since the eigenstates of H_J always contain an even number of kinks, the dispersion relation is determined by a set of two-parameters: the momenta of the even number of kinks whose scattering provides the excitation. There are no bound states of kinks.

2.2 Finite size antiferromagnetic Heisenberg chains

Let us now turn to the computation of the spectrum of finite size quantum antiferromagnetic chains by exact diagonalization. We shall see that already for very small chains, the spectrum is well described by the Bethe ansatz solution in the thermodynamic limit. Furthermore, an intuitive picture of the ground state and of the lowest lying excitations of the strongly coupled two-flavor lattice Schwinger model emerges, due to the mapping of the gauge model onto the spin chain $-$ see section 3.

The states of an antiferromagnetic chain are classified according to the quantum numbers of spin, third spin component, energy and momentum $|S, S^3, E, p>$. For a 4 site chain the momenta allowed for the states are: $0, \frac{\pi}{2}, \frac{3\pi}{2}$ mod 2π. The ground state is

$$|g.s.>=|0,0,-3J,0>=\frac{1}{\sqrt{12}}(2|\uparrow\downarrow\uparrow\downarrow>+2|\downarrow\uparrow\downarrow\uparrow>-|\uparrow\uparrow\downarrow\downarrow>-|\uparrow\downarrow\downarrow\uparrow>-|\downarrow\downarrow\uparrow\uparrow>-|\downarrow\uparrow\uparrow\downarrow>) \quad . \tag{2.75}$$

This state is P-parity even. In fact, by the definition of P-parity given in Eq.(3.178), the P-parity inverted state (2.75) is obtained by reverting the order of the spins in each vector $|\ldots>$ appearing in (2.75), e.g. $|\downarrow\downarrow\uparrow\uparrow>\stackrel{P}{\longrightarrow}|\uparrow\uparrow\downarrow\downarrow>$.

The λ's associated to the ground state (solution of the Bethe ansatz equations (2.29)) are $\lambda_1 = -\frac{1}{2\sqrt{3}}$ and $\lambda_2 = \frac{1}{2\sqrt{3}}$. There is also an excited singlet

$$|0,0,-J,\pi>=\frac{1}{\sqrt{4}}(|\downarrow\downarrow\uparrow\uparrow>-|\downarrow\uparrow\uparrow\downarrow>-|\uparrow\downarrow\downarrow\uparrow>+|\uparrow\uparrow\downarrow\downarrow>) \quad . \tag{2.76}$$

It is P-even, so that it is a $S^P = 0^+$ excitation, with the same quantum numbers (the isospin is replaced by the spin) of the lowest lying singlet excitation of the strongly coupled Schwinger model discussed by Coleman [26]. The state (2.76) also coincides with the excited singlet described by the configuration (2.42). It has only two complex λ's which arrange themselves in a string approximately of length 2, $\lambda_1 = -\lambda_2 = i\sqrt{\frac{\sqrt{481}-17}{8}}$ and there are two holes with $Q_1^{(h)} = -\frac{1}{2}$ and $Q_2^{(h)} = \frac{1}{2}$.

There are also three excited triplets, whose highest weight states are

$$|1,1,-J,\frac{\pi}{2}> \; = \; \frac{1}{\sqrt{4}}(|\downarrow\uparrow\uparrow\uparrow>+i|\uparrow\downarrow\uparrow\uparrow>-|\uparrow\uparrow\downarrow\uparrow>-i|\uparrow\uparrow\uparrow\downarrow>) \tag{2.77}$$

$$|1,1,-2J,\pi> \; = \; \frac{1}{\sqrt{4}}(|\downarrow\uparrow\uparrow\uparrow>-|\uparrow\downarrow\uparrow\uparrow>+|\uparrow\uparrow\downarrow\uparrow>-|\uparrow\uparrow\uparrow\downarrow>) \tag{2.78}$$

$$|1,1,-J,\frac{3\pi}{2}> \; = \; \frac{1}{\sqrt{4}}(|\downarrow\uparrow\uparrow\uparrow>-i|\uparrow\downarrow\uparrow\uparrow>-|\uparrow\uparrow\downarrow\uparrow>+i|\uparrow\uparrow\uparrow\downarrow>) \quad . \tag{2.79}$$

Among these, only the non-degenerate state with the lowest energy has a well defined P-parity (2.78). It is a $S^P = 1^-$ like the lowest lying triplet of the two-flavor strongly coupled Schwinger model. The degenerate states can be always combined to form a P-odd state.

We thus see that within the states in a given configuration there is always a representative state with well defined parity, the others are degenerate and can be used to construct state of well defined energy and parity. Moreover the parity of the representative states (with respect to the parity of the ground state) is the same of the one of the lowest-lying Schwinger model excitations in strong coupling.

All the triplets in (2.79) have one real λ and two holes; they can be associated with the family of triplets (2.43). In table (1) we summarize the triplet λ's and $Q^{(h)}$'s. The spectrum exhibits also

Table 1: Triplet internal quantum numbers

TRIPLET	λ	$Q_1^{(h)}$	$Q_2^{(h)}$
$\vert 1,1,-J,\frac{\pi}{2}>$	$\frac{1}{2}$	-1	0
$\vert 1,1,-2J,\pi>$	0	-1	1
$\vert 1,1,-J,\frac{3\pi}{2}>$	$-\frac{1}{2}$	0	1

a quintet, whose highest weight state is

$$|2,2,0,0>=|\uparrow\uparrow\uparrow\uparrow> \tag{2.80}$$

We report in fig.(2) the spectrum of the 4 sites chain.

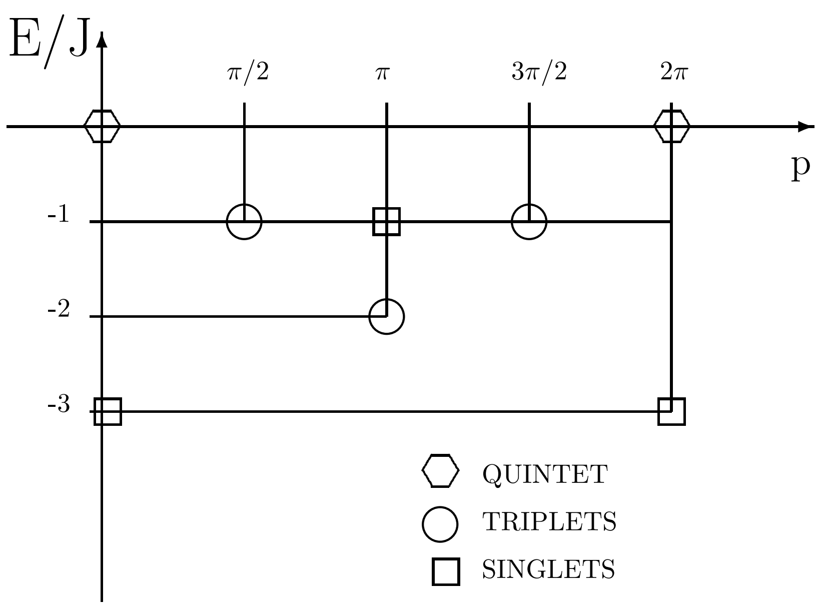

Figure 2: Four sites chain spectrum

Let us analize the spectrum of the 6 site antiferromagnetic chain. The momenta allowed for the states are now $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$ mod 2π. The ground state is

$$|g.s.> \;=\; |0,0,-\frac{J}{2}(5+\sqrt{13}),\pi>=\frac{1}{\sqrt{26-6\sqrt{13}}}\{|\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow> -|\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow>$$

$$
\begin{aligned}
&+ \frac{1-\sqrt{13}}{6}(|\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow> -|\uparrow\downarrow\uparrow\downarrow\downarrow\uparrow> +|\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow> -|\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow> +|\downarrow\downarrow\uparrow\uparrow\downarrow\uparrow> -|\downarrow\uparrow\uparrow\downarrow\uparrow\downarrow> \\
&- |\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow> +|\downarrow\uparrow\downarrow\uparrow\uparrow\downarrow> -|\uparrow\downarrow\uparrow\uparrow\downarrow\downarrow> +|\downarrow\uparrow\uparrow\downarrow\downarrow\uparrow> -|\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow> +|\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow>) \\
&+ \frac{4-\sqrt{13}}{3}(|\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow> -|\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow> +|\uparrow\downarrow\downarrow\downarrow\uparrow\uparrow> -|\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow> +|\downarrow\downarrow\uparrow\uparrow\uparrow\downarrow> -|\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow>)\}
\end{aligned}
\quad (2.81)
$$

This state is odd under P-parity. The spectrum of the six sites chain is reported in fig.(3). There are 9 triplets in the spectrum. In [27] it was already pointed out that the number of lowest lying triplets for a finite system with N sites is $N(N+2)/8$, so for $N=6$ there are 6 lowest lying triplet states. In order to identify these 6 states among the 9 that are exhibited by the spectrum of fig.(3), it is necessary to compute their λ's and their Q's. In this way in fact, We can find out which are the triplets characterized by two holes and thus belonging to the triplet of type (2.43). In table (2) we report the internal quantum numbers of the lowest lying triplets. The $Q^{(h)}$'s vary in the segment $[-\frac{3}{2},\frac{3}{2}]$. The highest weight state of the triplet of zero momentum and energy $-(J/2)(5+\sqrt{5})$ reads

$$
\begin{aligned}
|0,0,-\frac{J}{2}(5+\sqrt{5}),0> \;=\;& \frac{1}{\sqrt{45-15\sqrt{5}}}\{\frac{-3+\sqrt{5}}{2}(|\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow> +|\downarrow\uparrow\uparrow\uparrow\uparrow\downarrow> +|\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow> +|\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow> \\
&+ |\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow> +|\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow>) \\
&+ (|\downarrow\uparrow\downarrow\uparrow\uparrow\uparrow> +|\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow> +|\downarrow\uparrow\uparrow\uparrow\downarrow\uparrow> +|\uparrow\uparrow\uparrow\downarrow\uparrow\downarrow> +|\uparrow\uparrow\downarrow\uparrow\downarrow\uparrow> +|\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow>) \\
&+ (1-\sqrt{5})(|\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow> +|\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow> +|\uparrow\downarrow\uparrow\uparrow\downarrow\uparrow>)\} \quad .
\end{aligned}
\quad (2.82)
$$

One can get the triplet of energy $-(J/2)(5-\sqrt{5})$ from (2.82) by changing $\sqrt{5} \to -\sqrt{5}$. As can be explicitly checked from (2.82), the two non-degenerate triplets of zero momentum are then P-parity even, namely they have opposite parity with respect to that of the ground state, as it happens for the lowest lying triplet excitations of the two-flavor Schwinger model. For what concerns the degenerate triplets of momenta $\pi/3$ and $5\pi/3$ (or $2\pi/3$ and $4\pi/3$) they do not have definite P-parity, but it is always possible to take a linear combination of them with parity opposite to the ground state.

Table 2: Triplet internal quantum numbers

TRIPLET	λ_1	λ_2	$Q_1^{(h)}$	$Q_2^{(h)}$
$\vert 1,1,-\frac{5+\sqrt{5}}{2}J,0>$	$-\sqrt{\frac{5-2\sqrt{5}}{20}}$	$\sqrt{\frac{5-2\sqrt{5}}{20}}$	$-\frac{1}{2}$	$\frac{1}{2}$
$\vert 1,1,-\frac{5-\sqrt{5}}{2}J,0>$	$-\sqrt{\frac{5+2\sqrt{5}}{20}}$	$\sqrt{\frac{5+2\sqrt{5}}{20}}$	$-\frac{3}{2}$	$\frac{3}{2}$
$\vert 1,1,-\frac{5}{2}J,\frac{\pi}{3}>$	$-\frac{\sqrt{3}+\sqrt{\pi}}{8}$	$-\frac{\sqrt{3}-\sqrt{\pi}}{8}$	$-\frac{3}{2}$	$\frac{1}{2}$
$\vert 1,1,-\frac{7+\sqrt{17}}{4}J,\frac{2\pi}{3}>$	$\frac{-2\sqrt{3}-\sqrt{-2+2\sqrt{17}}}{2+2\sqrt{17}}$	$\frac{-2\sqrt{3}+\sqrt{-2+2\sqrt{17}}}{2+2\sqrt{17}}$	$-\frac{3}{2}$	$-\frac{1}{2}$
$\vert 1,1,-\frac{7+\sqrt{17}}{4}J,\frac{4\pi}{3}>$	$\frac{2\sqrt{3}-\sqrt{-2+2\sqrt{17}}}{2+2\sqrt{17}}$	$\frac{2\sqrt{3}+\sqrt{-2+2\sqrt{17}}}{2+2\sqrt{17}}$	$\frac{1}{2}$	$\frac{3}{2}$
$\vert 1,1,-\frac{5}{2}J,\frac{5\pi}{3}>$	$\frac{\sqrt{3}-\sqrt{\pi}}{8}$	$\frac{\sqrt{3}+\sqrt{\pi}}{8}$	$-\frac{1}{2}$	$\frac{3}{2}$

The remaining three triplets in fig.(3) have no real λ's and are characterized by a string of length 2 and four holes for $Q=-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2}$, *i.e.* do not belong to the type (2.43). More precisely, two triplets have a string approximately of length 2, due to the finite size of the system, while the triplet

of momentum π has a string exactly of length 2. In table (3) we summarize the quantum numbers of these triplets.

Table 3: Four holes triplet internal quantum numbers

TRIPLET	λ_1	λ_2
$\|1,1,-\frac{7-\sqrt{17}}{4}J,\frac{2\pi}{3}>$	$\frac{2\sqrt{3}-i\sqrt{2+2\sqrt{17}}}{-2+2\sqrt{17}}$	$\frac{2\sqrt{3}+i\sqrt{2+2\sqrt{17}}}{-2+2\sqrt{17}}$
$\|1,1,-J,\pi>$	$-\frac{i}{2}$	$\frac{i}{2}$
$\|1,1,-\frac{7-\sqrt{17}}{4}J,\frac{4\pi}{3}>$	$\frac{2\sqrt{3}+i\sqrt{2+2\sqrt{17}}}{2-2\sqrt{17}}$	$\frac{2\sqrt{3}-i\sqrt{2+2\sqrt{17}}}{2-2\sqrt{17}}$

In fig.(3) it is shown that the spectrum exhibits five singlet states. The lowest lying state at momentum π is the ground state. Then there are three excited singlets characterized by the configuration with two holes (2.42), *i.e.* they have one real λ and a string of length almost 2. In table (4) we summarize their quantum numbers. Among these singlets, those which are not degenerate, have P-parity equal to that of the ground state (odd) as it happens in the two-flavor Schwinger model. The non- singlet in fact reads

$$\begin{aligned}|0,0,-3J,0> &= \frac{1}{\sqrt{12}}\{|\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow> +|\uparrow\downarrow\uparrow\downarrow\downarrow\uparrow> +|\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow> +|\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow> +|\downarrow\downarrow\uparrow\uparrow\downarrow\uparrow> +|\downarrow\uparrow\uparrow\downarrow\uparrow\downarrow> \\ &- |\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow> -|\downarrow\uparrow\downarrow\uparrow\uparrow\downarrow> -|\uparrow\downarrow\uparrow\uparrow\downarrow\downarrow> -|\downarrow\uparrow\uparrow\downarrow\downarrow\uparrow> -|\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow> -|\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow>\} \quad . \end{aligned} \tag{2.83}$$

The degenerate singlets are again not eigenstates of the P-parity, but it is always possible to take a linear combination of them with the a P-parity that coincides with that of the representative state (2.83) of the configuration.

Table 4: Singlet internal quantum numbers

SINGLET	λ	λ_S	$Q_1^{(h)}$	$Q_2^{(h)}$
$\|0,0,-3J,0>$	0	0	-1	1
$\|0,0,-2J,\frac{\pi}{3}>$	$-\frac{\sqrt{3}+2\sqrt{6}}{14}$	$\frac{-2+3\sqrt{2}}{\sqrt{3}(4+\sqrt{2})}$	0	1
$\|0,0,-2J,\frac{5\pi}{3}>$	$\frac{\sqrt{3}+2\sqrt{6}}{14}$	$\frac{2-3\sqrt{2}}{\sqrt{3}(4+\sqrt{2})}$	-1	0

The remaining singlet $|0,0,-\frac{5-\sqrt{13}}{2}J,\pi>$ it is not of the type (2.42). It is characterized by a string approximately of length 3 with $\lambda_{1,1}=i\sqrt{\frac{5+2\sqrt{13}}{12}}$, $\lambda_{2,1}=0$ and $\lambda_{3,1}=-i\sqrt{\frac{5+2\sqrt{13}}{12}}$.

Even in finite systems very small like the 4 and 6 sites chains, the "string hypothesis" is a very good approximation and it allows us to classify and distinguish among states with the same spin.

The ground state of the antiferromagnetic Heisenberg chain with N sites is a linear combination of all the $\left(\begin{array}{c} N \\ \frac{N}{2} \end{array}\right)$ states with $\frac{N}{2}$ spins up and $\frac{N}{2}$ spins down. These states group themselves into

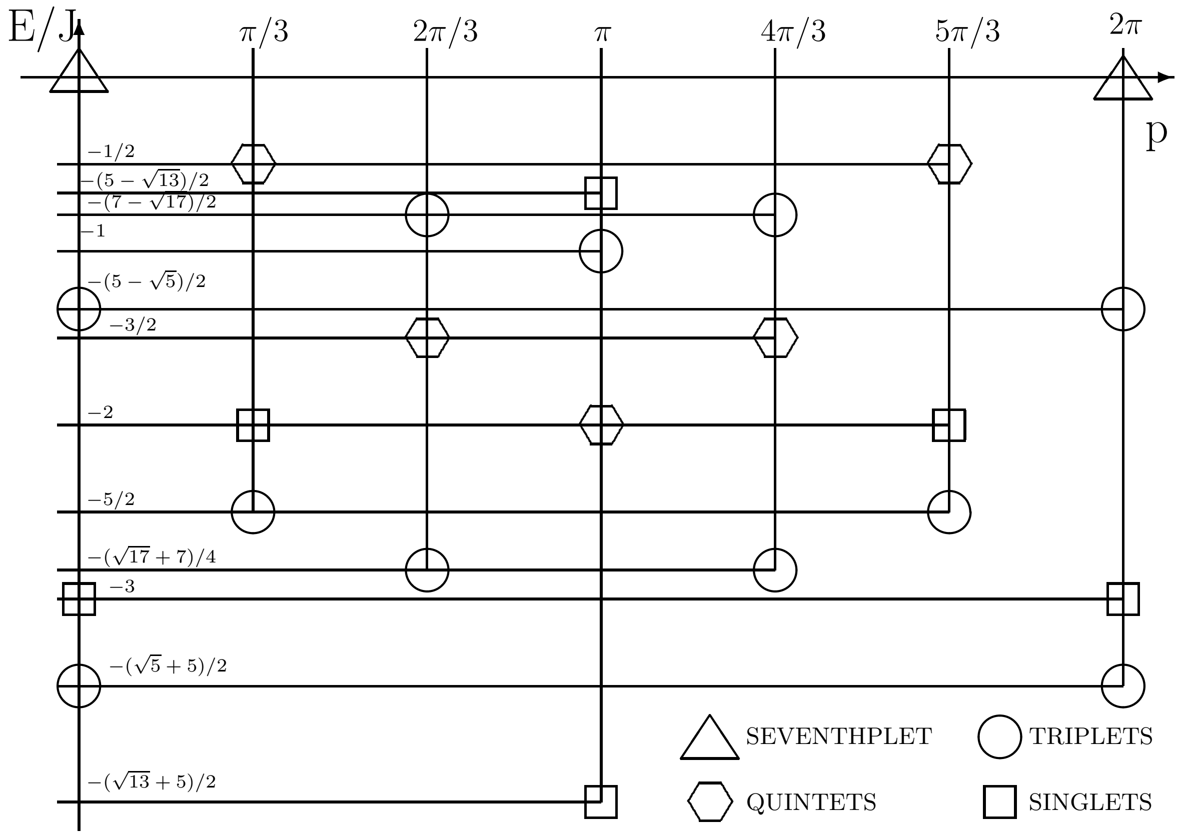

Figure 3: Six sites chain spectrum

sets with the same coefficient in the linear combination according to the fact that the ground state is translationally invariant (with momentum 0 (π) for $\frac{N}{2}$ even (odd)), it is an eigenstate of P-parity and it is invariant under the exchange of up with down spins. The states belonging to the same set have the same number of domain walls, which ranges from N, for the two Néel states, to 2 for the states with $\frac{N}{2}$ adjacent spins up and $\frac{N}{2}$ adjacent spins down.

The ground state of the 8 sites chain is

$$|g.s.>= \frac{1}{\sqrt{\mathcal{N}}}(|\psi_8> + \alpha|\psi_6^{(1)}> + \beta|\psi_6^{(2)}> + \gamma|\psi_4^{(1)}> + \delta|\psi_4^{(2)}> + \epsilon|\psi_4^{(3)}> + \zeta|\psi_2>) \tag{2.84}$$

where

$$|\psi_8>=|\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow> + |\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow> \tag{2.85}$$
$$|\psi_6^{(1)}>=|\uparrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow> + |\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow> + translated \quad states \tag{2.86}$$
$$|\psi_6^{(2)}>=|\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow> + |\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow\downarrow\uparrow> + translated \quad states \tag{2.87}$$
$$|\psi_4^{(1)}>=|\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow\downarrow> + translated \quad states \tag{2.88}$$
$$|\psi_4^{(2)}>=|\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow> + |\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow\downarrow\uparrow> + translated \quad states \tag{2.89}$$
$$|\psi_4^{(3)}>=|\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\uparrow\downarrow> + |\downarrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow> + translated \quad states \tag{2.90}$$
$$|\psi_2>=|\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow> + translated \quad states\ . \tag{2.91}$$

By direct diagonalization one gets

$$\alpha = -0.412773 \tag{2.92}$$

$$\beta = 0.344301 \tag{2.93}$$
$$\gamma = 0.226109 \tag{2.94}$$
$$\delta = -0.087227 \tag{2.95}$$
$$\epsilon = 0.136945 \tag{2.96}$$
$$\zeta = 0.018754 \tag{2.97}$$
$$\mathcal{N} = 2 + 16\alpha^2 + 8\beta^2 + 4\gamma^2 + 16\delta^2 + 16\epsilon^2 + 8\zeta^2 = 6.30356 \quad . \tag{2.98}$$

The energy of the ground state is

$$E_{g.s.} = -5.65109J \quad . \tag{2.99}$$

Eq.(2.99) differs only by 1.8% from the thermodynamic limit expression $E_{g.s.} = -8\ln 2 = -5.54518$. Moreover also the correlation function of distance 2 Eq.(2.101) computed for the 8 sites chain is $G(2) = 0.1957N$, value which is 7% higher than the exact answer Eq.(2.102).

In the analysis of finite size systems we were able to find the coefficient β of the first set of states containing $N-2$ domain walls in the ground state. These states are obtained interchanging two adjacent spins in the Néel states. The β is for a generic chain of N-sites

$$\beta = \frac{N + 2E_{g.s.}}{N} = 1 - 2\ln 2 \quad . \tag{2.100}$$

2.3 Spin-spin correlators

The explicit computation of spin-spin correlations is far from being trivial since the correlator $G(r) =< g.s.|\vec{S}_0 \cdot \vec{S}_r|g.s. >$ is not known for arbitrary lattice separations r. For $r = 2$ it was computed by M. Takahashi [25] in his perturbative analysis of the half filled Hubbard model in one dimension. For $r > 2$ no exact numerical values of $G(r)$ are known. In [28] were given two representations of $G(r)$, while in [29, 30] the exact asymptotic ($r \to \infty$) expression of $G(r)$ was derived.

In order to explicitly compute the second order energies Eq.(3.211) and Eq.(3.212) one has to evaluate the correlation function

$$G(2) = \frac{1}{N}\sum_{x=1}^{N} < g.s.|\vec{S}_x \cdot \vec{S}_{x+2}|g.s > \tag{2.101}$$

which has been exactly computed in [25] and is given by

$$G(2) = \frac{1}{4}(1 - 16\ln 2 + 9\zeta(3)) = 0.1820 \quad . \tag{2.102}$$

In the following we shall show how the knowledge of this correlator allows one to compute explicitly the first three "emptiness formation probabilities", used in Ref. [28] in the study of the Heisenberg chain correlators, $G(r)$. The isotropy of the Heisenberg model implies that

$$\sum_{x=1}^{N} < g.s.|\vec{S}_x \cdot \vec{S}_{x+2}|g.s. >= 3\sum_{x=1}^{N} < g.s.|S^3_x \cdot S^3_{x+2}|g.s. > \quad . \tag{2.103}$$

Let us introduce the probability P_3 for finding three adjacent spins in a given position in the Heisenberg antiferromagnetic vacuum. Taking advantage of the isotropy of the Heisenberg model ground state and of its translational invariance, it is easy to see that the correlator (2.103) can be written in terms of the P_3's as

$$\sum_{x=1}^{N} < g.s.|S^3_x \cdot S^3_{x+2}|g.s >= N\,\frac{1}{4}\,2\,\left(\,P_3(\uparrow\uparrow\uparrow) + P_3(\uparrow\downarrow\uparrow) - P_3(\uparrow\uparrow\downarrow) - P_3(\downarrow\uparrow\uparrow)\,\right) \quad . \tag{2.104}$$

The factor 2 appears in (2.104) due again to the isotropy of the Heisenberg model: the probability of a configuration and of the configuration rotated by π around the chain axis, are the same.

In [28] the so called "emptiness formation probability" $P(x)$ was introduced.

$$P(x) =< g.s.| \prod_{j=1}^{x} P_j |g.s. > \quad , \tag{2.105}$$

where

$$P_j = \frac{1}{2}(\sigma_j^3 + 1) \tag{2.106}$$

and σ_j^3 is the Pauli matrix. $P(x)$ determines the probability of finding x adjacent spins up in the antiferromagnetic vacuum. One gets

$$\begin{aligned} P(\uparrow\uparrow\uparrow) &= P(3) & (2.107)\\ P(\uparrow\downarrow\uparrow) &= P(1) - 2P(2) + P(3) & (2.108)\\ P(\downarrow\uparrow\uparrow) &= P(\uparrow\uparrow\downarrow) = P(2) - P(3) & (2.109) \end{aligned}$$

so that Eq.(2.101) reads

$$G(2) = 2P(3) - 2P(2) + \frac{1}{2}P(1) \quad . \tag{2.110}$$

Using the exact value the correlator $G(2)$ computed in [25] from (2.110) and from the known values of $P(2)$ and $P(1)$ given in [28]

$$\begin{aligned} P(1) &= \frac{1}{2} & (2.111)\\ P(2) &= \frac{1}{3}(1 - \ln 2) & (2.112)\\ & & (2.113) \end{aligned}$$

one gets

$$P(3) = \frac{1}{3}(1 - 7\ln 2) + \frac{9}{8}\zeta(3) \quad . \tag{2.114}$$

For the general emptiness formation probability $P(x)$ of the antiferromagnetic Heisenberg chain, an integral representation was given in [28], but, to our knowledge, the exact value of $P(3)$ (2.114) was not known.

We illustrate now the computation of a spin-spin correlator which appears in the mass spectrum of the two-flavor lattice Schwinger model

$$\begin{aligned} < g.s.|\vec{V} \cdot \vec{V}|g.s. >= \sum_{x,y=1}^{N} (< g.s.|(\vec{S}_x \cdot \vec{S}_y)(\vec{S}_{x+1} \cdot \vec{S}_{y+1})|g.s. > \\ - < g.s.|(\vec{S}_x \cdot \vec{S}_{y+1})(\vec{S}_{x+1} \cdot \vec{S}_y)|g.s. >) - \sum_{x=1}^{N} < g.s.|\vec{S}_x \cdot \vec{S}_{x+1}|g.s. > \quad . \end{aligned} \tag{2.115}$$

It is possible to extract a numerical value from Eq.(2.115) only within the random phase approximation [25, 31]. For this purpose it is first convenient to rewrite the unconstrained sum over the sites x and y as a sum where all the four spins involved in the VEV's lie on different sites,

$$\begin{aligned} < g.s.|\vec{V} \cdot \vec{V}|g.s. > &= \sum_{\substack{y\neq x \\ y\neq x\pm 1}} (< g.s.|(\vec{S}_x \cdot \vec{S}_y)(\vec{S}_{x+1} \cdot \vec{S}_{y+1})|g.s. > \\ &- < g.s.|(\vec{S}_x \cdot \vec{S}_{y+1})(\vec{S}_{x+1} \cdot \vec{S}_y)|g.s. >) + \frac{3}{8}N \\ &- \frac{1}{2}\sum_{x=1}^{N} < g.s.|\vec{S}_x \cdot \vec{S}_{x+1}|g.s. > - \sum_{x=1}^{N} < g.s.|\vec{S}_x \cdot \vec{S}_{x+2}|g.s. > \end{aligned} \tag{2.116}$$

and then factorize the four spin operators in Eq.(2.116) as

$$
\begin{aligned}
< g.s.|\vec{V}\cdot\vec{V}|g.s.> &= N\sum_{r=2}^{\infty}(< g.s.|\vec{S}_0\cdot\vec{S}_r|g.s.>^2 - < g.s.|\vec{S}_0\cdot\vec{S}_{r+1}|g.s.>< g.s.|\vec{S}_1\cdot\vec{S}_r|g.s.>) \\
&+ \frac{3}{8}N - \frac{1}{2}\sum_{x=1}^{N} < g.s.|\vec{S}_x\cdot\vec{S}_{x+1}|g.s.> - \sum_{x=1}^{N} < g.s.|\vec{S}_x\cdot\vec{S}_{x+2}|g.s.> \quad (2.117)
\end{aligned}
$$

Of course, Eq.(2.117) provides an answer larger than the exact result; terms such as $<(\ldots)(\ldots)>$ yield negative contributions which are eliminated once one factorizes them in the form $<(...)><(...)>$. This is easily checked also by direct computation on finite size systems.

The spin-spin correlation functions G(r) are exactly known for $r = 1, 2$. For the spin-spin correlation functions $G(r)$ up to a distance of $r = 30$ the results are reported in table (5) [29, 32].

Table 5: Spin-spin correlation functions

r	$G(r)$	r	$G(r)$
1	-0.4431	16	0.0305
2	0.1821	17	-0.0296
3	-0.1510	18	0.0274
4	0.1038	19	-0.0267
5	-0.0925	20	0.0249
6	0.0731	21	-0.0242
7	-0.0671	22	0.0228
8	0.0567	23	-0.0223
9	-0.0532	24	0.0211
10	0.0465	25	-0.0206
11	-0.0442	26	0.0196
12	0.0395	27	-0.0193
13	-0.0379	28	0.0183
14	0.0344	29	-0.0181
15	-0.0332	30	0.0172

For $r > 30$, one may write [29]

$$
\begin{aligned}
G(r) &= \frac{3}{4}\sqrt{\frac{2}{\pi^3}}\frac{1}{r\sqrt{g(r)}}[1 - \frac{3}{16}g(r)^2 + \frac{156\zeta(3)-73}{384}g(r)^3 + O(g(r)^4) - \\
&\quad \frac{0.4}{2r}((-1)^r + 1 + O(g(r)) + O(\frac{1}{r^2})] \quad (2.118)
\end{aligned}
$$

with $g(r)$ satisfying

$$g(r) = \frac{1}{C(r)}(1 + \frac{1}{2}g(r)\ln(g(r))) \quad (2.119)$$

and

$$C(r) = \ln(2\sqrt{2\pi}e^{\gamma+1}r) \quad . \quad (2.120)$$

Eq.(2.119) may be solved by iteration. To the lowest order in $\frac{1}{C}$ one finds

$$g(r) \approx \frac{1}{C(r)} - \frac{1}{C(r)^2}\ln C(r) \quad . \quad (2.121)$$

Inserting (2.121) in Eq.(2.118) leads to

$$G(r) \approx \sqrt{2}\pi^3 \frac{1}{r}\sqrt{C(r)}[1 + \frac{1}{4C(r)} \ln C(r)] + O(\frac{1}{C(r)^2}) \quad . \tag{2.122}$$

Inserting Eq.(2.122) in (2.117), one finally gets

$$< g.s.|\vec{V} \cdot \vec{V}|g.s. >= 0.3816N \tag{2.123}$$

Last, we report the following exact three spin correlators that have been used in the determination of the mass spectrum of the two-flavor lattice Schwinger model

$$< g.s.|\sum_{x=1}^{N} S_x^3 S_{x+1}^3 S_{x+2}^3|g.s. > \; = \; 0 \tag{2.124}$$

$$< g.s.|\sum_{x=1}^{N} S_x^+ S_{x+1}^- S_{x+2}^3|g.s. > \; = \; 0 \tag{2.125}$$

$$< g.s.|\sum_{x=1}^{N} S_x^- S_{x+1}^+ S_{x+2}^3|g.s. > \; = \; 0 \tag{2.126}$$

$$< g.s.|\sum_{x=1}^{N} S_x^+ S_{x+1}^3 S_{x+2}^-|g.s. > \; = \; 0 \tag{2.127}$$

$$< g.s.|\sum_{x=1}^{N} S_x^- S_{x+1}^3 S_{x+2}^+|g.s. > \; = \; 0 \tag{2.128}$$

So, on a spin singlet, not only the VEV of $\sum_{x=1}^{N} S_x^3$ is zero, but also every VEV with an odd number of S^3.

2.4 $SU(\mathcal{N})$ quantum antiferromagnetic chains

It is our purpose to introduce now spin-1/2 antiferromagnetic Heisenberg chains where "spins" are generators of the $SU(\mathcal{N})$ group. In the limit $\mathcal{N} = 2$ one has the usual antiferromagnetic Heisenberg chain discussed in the previous section. An $U(\mathcal{N})$ spin-1/2 quantum antiferromagnetic chain is described by the Hamiltonian

$$H_J^{U(\mathcal{N})} = J \sum_{x=1}^{N} S_{ab}(x) S_{ba}(x+1) \tag{2.129}$$

where $S_{ab}(x)$ $a, b = 1, \ldots, \mathcal{N}$ are the generators of $U(\mathcal{N})$ satisfying the Lie algebra

$$[S_{ab}(x), S_{cd}(y)] = (S_{ad}(x)\delta_{bc} - S_{cb}(x)\delta_{ad})\delta_{xy} \tag{2.130}$$

and they can be conventionally represented by fermion bilinear operators

$$S_{ab}(x) = \psi_{ax}^\dagger \psi_{bx} - \frac{\delta_{ab}}{2} \quad . \tag{2.131}$$

The representation of the algebra on each site is fixed by specifying the fermion number occupation

$$\rho(x) = \sum_{a=1}^{\mathcal{N}} S_{aa}(x) \quad . \tag{2.132}$$

By fulfilling the global neutrality condition $\sum_{x=1}^{N} \rho(x) = 0$, one may choose

$$\sum_{a=1}^{\mathcal{N}} \psi_{ax}^\dagger \psi_{ax} = \begin{cases} m & x \;\; \text{even} \\ \mathcal{N} - m & x \;\; \text{odd} \end{cases} \tag{2.133}$$

or viceversa the opposite choice for x even-odd. Eq.(2.133) restricts on each site the Hilbert space to a representation with Young tableau of m rows for x even and $\mathcal{N}-m$ rows for x odd. For each site x $\rho(x)$ is the generator of the $U(1)$ subgroup of $U(\mathcal{N})$.

Let us use the basis $T^\alpha = (T^\alpha)^*$, $\alpha = 1,\ldots,\mathcal{N}^2-1$, of the Lie algebra of $SU(\mathcal{N})$ in the fundamental representation such that $tr(T^\alpha T^\beta) = \delta^{\alpha\beta}/2$ and $\left[T^\alpha, T^\beta\right] = if^{\alpha\beta\gamma}T^\gamma$, where $f^{\alpha\beta\gamma}$ are the structure constants. By means of

$$T^\alpha_{ab}T^\alpha_{cd} = \frac{1}{2}\delta_{ad}\delta_{bc} - \frac{1}{2\mathcal{N}}\delta_{ab}\delta_{cd} \tag{2.134}$$

and redefining the group generators

$$S^\alpha_x = \psi^\dagger_{ax}T^\alpha_{ab}\psi_{bx} \tag{2.135}$$

one can rewrite the Hamiltonian (2.129) as

$$H_J^{U(\mathcal{N})} = J\sum_{x=1}^{N}\rho(x)\rho(x+1) + H_J^{SU(\mathcal{N})} \tag{2.136}$$

where

$$H_J^{SU(\mathcal{N})} = J\sum_{x=1}^{N} S^\alpha_x S^\alpha_{x+1} \tag{2.137}$$

is the Hamiltonian of an $SU(\mathcal{N})$ quantum antiferromagnet. From Eq.(2.136) it is clear that by fixing $\rho(x)$ $H_J^{U(\mathcal{N})}$ is reduced to $H_J^{SU(\mathcal{N})}$.

The representations of the $SU(\mathcal{N})$ spins relevant for the relationship of the spin Hamiltonians (2.137) with the Schwinger models are different when $\mathcal{N}$ is even or odd. When $\mathcal{N}$ is even we shall consider the representation on each site with Young tableau of one column and $\mathcal{N}/2$ rows so that $\rho(x) = 0$. When $\mathcal{N}$ is odd we shall consider on one sublattice the representation with Young tableau of one column and $(\mathcal{N}+1)/2$ rows and of one column and $(\mathcal{N}-1)/2$ rows on the other sublattice.

As an explicit example let us consider the case $\mathcal{N} = 3$. The generators T^α of the $SU(3)$ group in Eq.(2.135) are the Gell Mann matrices. By denoting u,d and s the three flavors, the eight spin operators S^α_x read

$$\begin{aligned}
S^1_x &= \psi^\dagger_{ux}\psi_{dx} + \psi^\dagger_{dx}\psi_{ux} && (2.138)\\
S^2_x &= -i\psi^\dagger_{ux}\psi_{dx} + i\psi^\dagger_{dx}\psi_{ux} && (2.139)\\
S^3_x &= \psi^\dagger_{ux}\psi_{ux} - \psi^\dagger_{dx}\psi_{dx} && (2.140)\\
S^4_x &= \psi^\dagger_{ux}\psi_{sx} + \psi^\dagger_{sx}\psi_{ux} && (2.141)\\
S^5_x &= -i\psi^\dagger_{ux}\psi_{sx} + i\psi^\dagger_{sx}\psi_{ux} && (2.142)\\
S^6_x &= \psi^\dagger_{dx}\psi_{sx} + \psi^\dagger_{sx}\psi_{dx} && (2.143)\\
S^7_x &= -i\psi^\dagger_{dx}\psi_{sx} + i\psi^\dagger_{sx}\psi_{dx} && (2.144)\\
S^8_x &= \frac{1}{\sqrt{3}}(\psi^\dagger_{ux}\psi_{ux} + \psi^\dagger_{dx}\psi_{dx} - 2\psi^\dagger_{sx}\psi_{sx}) && (2.145)
\end{aligned}$$

Let us find the ground state of the Hamiltonian (2.137) for a chain of two sites. Taking the representations with one particle on site 1 and two particles on site 2, the ground state with energy $E_{g.s.} = -16J/3$ reads

$$|g.s.> = \frac{1}{\sqrt{3}}(|u \begin{array}{c} d \\ s \end{array} > -|d \begin{array}{c} u \\ s \end{array} > +|s \begin{array}{c} u \\ d \end{array} >) \tag{2.146}$$

The state (2.146) is a singlet of $SU(3)$, *i.e.* it is annihilated by the Casimir $\vec{S}^2 = (\vec{S}_1 + \vec{S}_2)^2 = \sum_{\alpha=1}^{8}\left[(S^\alpha_1)^2 + (S^\alpha_2)^2 + 2S^\alpha_1 S^\alpha_2\right]$. If one chooses the representation with two particles on site 1 and one particle on site 2, the ground state degenerate with (2.146) is

$$|g.s.>' = \frac{1}{\sqrt{3}}(| \begin{array}{c} d \\ s \end{array} u > -| \begin{array}{c} u \\ s \end{array} d > +| \begin{array}{c} u \\ d \end{array} s >) \tag{2.147}$$

Diagonalizing the translation operator $\hat{T}$ with the Hamiltonian (2.137), one gets two degenerate ground states

$$|G.S.>^{\pm}= \frac{|g.s.> \pm |g.s.>'}{\sqrt{2}} \tag{2.148}$$

In Eq.(2.148) the linear combination with the + has momentum zero and the one with the − has momentum π. By studying this very simple example, one can infer that the thermodynamic limit $N \to \infty$ analysis for a generic $SU(\mathcal{N})$ chain is very involved.

Unfortunately, no analysis with a level of completeness such as the one given in [18] for the $SU(2)$ case does exist for a generic symmetry group $SU(\mathcal{N})$ with spins in the representation in which we are interested. In [33] $SU(\mathcal{N})$ antiferromagnetic models were solved for a particular spin representation such that the Hamiltonian (2.137) becomes

$$H_J^{SU(\mathcal{N})} = J \sum_{x=1}^{N} P_{xx+1} \tag{2.149}$$

with P_{xx+1} the operator that permutes whatever objects occupy sites x and $x+1$. The spectrum of Eq.(2.149) was shown to exhibit massless excitations.

A large $\mathcal{N}$ expansion approach has been performed in [34] for an $SU(\mathcal{N})$ antiferromagnetic chain characterized by spins living in a representation with Young tableaux of one row on one sublattice and $\mathcal{N}-1$ rows on the other sublattice. In the case of Young tableaux of one column it was found that the ground state is twofold degenerate and breaks translational and parity symmetry. Moreover the elementary excitations are massive non relativistic solitons in the large $\mathcal{N}$ limit with a mass of $O(\mathcal{N})$.

In [35] the Lieb-Shultz-Mattis theorem [36] was generalized to $SU(\mathcal{N})$ spin chains. The theorem proves that a half-integer-S spin chain with essentially any reasonably local Hamiltonian respecting translational and rotational symmetry either has zero gap or else has degenerate ground states spontaneously breaking translational and parity invariance. In [36] it was proved the existence of a unique ground state of the $SU(\mathcal{N})$ chains where $\mathcal{N}$ is even for spins in the antisymmetric $\mathcal{N}$ tensor representation. Under the assumption of a unique ground state which must be a $SU(\mathcal{N})$ singlet, an infinitesimal energy gap was found for all the representations of $SU(\mathcal{N})$ whose Young tableaux contain a number of boxes not divisible by $\mathcal{N}$. Of course the degeneracy of the ground state trivially implyes a zero gap.

Using the Lieb-Shultz-Mattis theorem we can state that when $\mathcal{N}$ is even the ground state of the Hamiltonian (2.137) is unique and there are gapless excitations for the spin representation with a Young tableau of one column and $\mathcal{N}/2$ rows. When $\mathcal{N}$ is odd the ground state is twofold degenerate, as we illustrated with the simple example of the two site $SU(3)$ chain. It is doubtful if there exist gapless excitations in this case and at present time we are investigating this problem.

3 The two-flavor lattice Schwinger model

The one-flavor lattice Schwinger model has been studied in [14]. The solution of the strong coupling problem is identical to solving a particular type of Ising system with long range interaction. We showed that the mass of the elementary excitations and the chiral condensate could be computed reliably from an extrapolation to weak coupling using Padé approximants. Our analysis improved previous ones [37] by taking careful account of all discrete symmetries of the continuum theory. We do not discuss here the one-flavor model, since it is mapped into the classical Ising spin model and it is out of the scope of these lectures.

We study the $SU(2)$-flavor lattice Schwinger model in the hamiltonian formalism using staggered fermions. The existence of the continuum internal isospin symmetry makes the model much more interesting than the one-flavor case; the spectrum is extremely richer, exhibiting also massless excitations and the chiral symmetry breaking pattern is completely different from the one-flavor case. We shall demonstrate [19, 15] that the strong coupling limit of the two-flavor lattice Schwinger

model is mapped onto an interesting quantum spin model − the one-dimensional spin-1/2 quantum Heisenberg antiferromagnet. The ground state of the antiferromagnetic chain has been known since many years [17] and its energy was computed in [23]; the complete spectrum as been determined by Faddeev and Takhtadzhyan [18] using the algebraic Bethe ansatz.

The two-flavor lattice Schwinger model with non-zero fermion mass m has been analysed in [38] in the limit of heavy fermions $m \gg e^2$; good agreement with the continuum theory has been found.

There are by now many hints at a correspondence between quantized gauge theories and quantum spin models, aimed at analyzing new phases relevant for condensed matter systems [5, 6, 7, 8, 9]. Recently Laughlin has argued that there is an analogy between the spectral data of gauge theories and strongly correlated electron systems [39]. Moreover, certain spin ladders have been shown to be related to the two-flavor Schwinger model [45].

The correspondence between the SU(2) flavor Schwinger model and the quantum Heisenberg antiferromagnetic chain provides a concrete computational scheme in which the issue of the correspondence between quantized gauge theories and quantum spin models may be investigated. Because of dimensionality of the coupling constant in (1+1)-dimensions the infrared behavior is governed by the strong coupling limit, and it is tempting to conjecture the existence of an exact correspondence between the infrared limits of the Heisenberg and two-flavor Schwinger models. We shall derive [19, 15] results which support this conjecture. For example the gapless modes in the spectra have identical quantum numbers; within the accurancy of the strong coupling limit, the gapped mode of the two-flavor Schwinger model was also identified in the spectrum of the Heisenberg model.

In this section we present a complete study [19, 15, 16] of the strong coupling limit of the two-flavor lattice Schwinger model. We firstly compute explicitly the masses of the excitations to the second order in the strong coupling expansion; this computation needs the knowledge of the spin-spin correlators of the quantum Heisenberg antiferromagnetic chain. The continuum massless two-flavor Schwinger model does exhibit neither an isoscalar $\langle\bar\psi\psi\rangle$ nor an isovector $\langle\bar\psi\sigma^a\psi\rangle$ chiral condensate, since this is forbidden by the Coleman theorem [22]. On the lattice these fermion condensates are zero to all the orders in the strong coupling expansion [15]. The pertinent non-zero chiral condensate is $< \overline{\psi}_L^{(2)}\overline{\psi}_L^{(1)}\psi_R^{(1)}\psi_R^{(2)} >$ and we computed its lattice expression up to the second order in the strong coupling expansion [15]. It should be noticed that, in abscence of gauge fields, the chiral condensate is zero, is different from zero only when the fermions are coupled to gauge fields. This can be viewed as the manifestation of the chiral anomaly in this model.

3.1 The model

The action of the $1+1$-dimensional electrodynamics with two charged Dirac spinor fields is

$$S = \int d^2x \left[\sum_{a=1}^{2} \overline{\psi}_a (i\gamma_\mu \partial^\mu + \gamma_\mu A^\mu)\psi_a - \frac{1}{4e_c^2} F_{\mu\nu}F^{\mu\nu} \right] \tag{3.150}$$

The theory has an internal $SU_L(2)\otimes SU_R(2)$-flavor isospin symmetry; the Dirac fields are an isodoublet whereas the electromagnetic field is an isosinglet. It is well known that in 1+1 dimensions there is no spontaneous breakdown of continuous internal symmetries, unless there are anomalies or the Higgs phenomenon occurs. Neither mechanism is possible in the two-flavor Schwinger model for the $SU_L(2)\otimes SU_R(2)$-symmetry: isovector currents do not develop anomalies and there are no gauge fields coupled to the isospin currents. The particles belong then to isospin multiplets. For what concerns the $U(1)$ gauge symmetry there is an Higgs phenomenon [41].

The action is invariant under the symmetry

$$SU_L(2)\otimes SU_R(2)\otimes U_V(1)\otimes U_A(1) \tag{3.151}$$

The group generators act on the fermion isodoublet to give

$$SU_L(2) \quad : \quad \psi_a(x) \longrightarrow (e^{i\theta_\alpha \frac{\sigma^\alpha}{2} P_L})_{ab}\ \psi_b(x)\ ,\ \overline{\psi_a}(x) \longrightarrow \overline{\psi_b}\ (x)(e^{-i\theta_\alpha \frac{\sigma^\alpha}{2} P_R})_{ba} \tag{3.152}$$

$$
\begin{aligned}
SU_R(2) &: \quad \psi_a(x) \longrightarrow (e^{i\theta_\alpha \frac{\sigma^\alpha}{2} P_R})_{ab}\ \psi_b(x)\ ,\ \overline{\psi_a}(x) \longrightarrow \overline{\psi_b}(x)\ (e^{-i\theta_\alpha \frac{\sigma^\alpha}{2} P_L})_{ba} && (3.153)\\
U_V(1) &: \quad \psi_a(x)\ \longrightarrow (e^{i\theta(x)\mathbf{1}})_{ab}\ \psi_b(x)\ ,\ \psi_a^\dagger(x) \longrightarrow \psi_b^\dagger(x)\ (e^{-i\theta(x)\mathbf{1}})_{ba} && (3.154)\\
U_A(1) &: \quad \psi_a(x) \longrightarrow (e^{i\alpha\gamma_5\mathbf{1}})_{ab}\ \psi_b(x)\ ,\ \psi_a^\dagger(x) \longrightarrow \psi_b^\dagger(x)\ (e^{-i\alpha\gamma_5\mathbf{1}})_{ba} \quad , && (3.155)
\end{aligned}
$$

where σ^α are the Pauli matrices, θ_α, $\theta(x)$ and α are real coefficients and

$$
P_L = \frac{1}{2}(1-\gamma_5)\ ,\ P_R = \frac{1}{2}(1+\gamma_5) \quad . \tag{3.156}
$$

At the classical level the symmetries (3.152−3.155) lead to conservation laws for the isovector, vector and axial currents

$$
\begin{aligned}
j^\mu_\alpha(x)_R &= \overline{\psi}_a(x)\gamma^\mu P_R(\frac{\sigma_\alpha}{2})_{ab}\psi_b(x) && (3.157)\\
j^\mu_\alpha(x)_L &= \overline{\psi}_a(x)\gamma^\mu P_L(\frac{\sigma_\alpha}{2})_{ab}\psi_b(x) && (3.158)\\
j^\mu(x) &= \overline{\psi}_a(x)\gamma^\mu \mathbf{1}_{ab}\psi_b(x) && (3.159)\\
j^\mu_5(x) &= \overline{\psi}_a(x)\gamma^\mu\gamma^5 \mathbf{1}_{ab}\psi_b(x) && (3.160)
\end{aligned}
$$

It is well known that at the quantum level the vector and axial currents cannot be simultaneously conserved, due to the anomaly phenomenon [43]. If the regularization is gauge invariant, so that the vector current is conserved, then the axial current acquires the anomaly which breaks the $U_A(1)$-symmetry

$$
\partial_\mu j^\mu_5(x) = 2\frac{e_c^2}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}(x) \tag{3.161}
$$

The isoscalar and isovector chiral condensates are zero due to the Coleman theorem [22]; in fact, they would break not only the $U_A(1)$ symmetry of the action, but also the continuum internal symmetry $SU_L(2)\otimes SU_R(2)$ down to $SU_V(2)$. There is, however, a $SU_L(2)\otimes SU_R(2)$ invariant operator, which is non-invariant under the $U_A(1)$-symmetry; it can acquire a non-vanishing VEV without violating Coleman's theorem and consequently may be regarded as a good order parameter for the $U_A(1)$-breaking. Its expectation value is given by [46, 42]

$$
< F > \equiv < \overline{\psi}_L^{(2)}\overline{\psi}_L^{(1)}\psi_R^{(1)}\psi_R^{(2)} > = (\frac{e^\gamma}{4\pi})^2\frac{2}{\pi}e_c^2 \quad . \tag{3.162}
$$

It describes a process in which two right movers are anihilated and two left movers are created. Note that F, being quadrilinear in the fields, is actually invariant under chiral rotations of $\pi/2$, namely under the discrete axial symmetry

$$
\psi_a(x) \to \gamma^5\psi_a(x) \quad \bar{\psi}_a(x) \to -\bar{\psi}_a(x)\gamma_5 \quad . \tag{3.163}
$$

As a consequence, this part of the chiral symmetry group is not broken by the non-vanishing VEV of F (3.162).

The lattice theory faithfully reproduces the pattern of symmetry breaking of the continuum theory; this happens even if on the lattice the $SU(2)$-flavor symmetry is not protected by the Coleman theorem. The isoscalar and isovector chiral condensates are zero also on the lattice, whereas the operator F acquires a non-vanishing VEV due to the coupling of left and right movers induced by the gauge field. The continuous axial symmetry is broken explicitly by the staggered fermion, but the discrete axial symmetry (3.163) remains.

The action (3.150) may be presented in the usual abelian bosonized form [47]. Setting

$$
:\overline{\psi}_a\gamma^\mu\psi_a := \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\Phi_a\ ,\ a=1,2 \quad , \tag{3.164}
$$

the electric charge density and the action read

$$
j_0 =: \psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2 := \frac{1}{\sqrt{\pi}}\partial_x(\Phi_1+\Phi_2) \tag{3.165}
$$

$$S=\int d^2x\left[\frac{1}{2}\partial_\mu\Phi_1\partial^\mu\Phi_1+\frac{1}{2}\partial_\mu\Phi_2\partial^\mu\Phi_2-\frac{e_c^2}{2\pi}(\Phi_1+\Phi_2)^2\right] \quad . \tag{3.166}$$

By changing the variables to

$$\Phi_+ = \frac{1}{\sqrt{2}}(\Phi_1+\Phi_2) \tag{3.167}$$

$$\Phi_- = \frac{1}{\sqrt{2}}(\Phi_1-\Phi_2) \quad , \tag{3.168}$$

one has

$$S=\int d^2x\left(\frac{1}{2}\partial_\mu\Phi_+\partial^\mu\Phi_+ +\frac{1}{2}\partial_\mu\Phi_-\partial^\mu\Phi_- -\frac{e_c^2}{\pi}\Phi_+^2\right) \quad . \tag{3.169}$$

The theory describes two scalar fields, one massive and one massless. Φ_+ is an isosinglet as evidenced from Eq.(3.165); its mass $m_S=\sqrt{\frac{2}{\pi}}e_c$ comes from the anomaly Eq.(3.161) [41]. Local electric charge conservation is spontaneously broken, but no Goldstone boson appears because the Goldstone mode may be gauged away. Φ_- represents an isotriplet; it has rather involved nonlinear transformation properties under a general isospin transformation. All three isospin currents can be written in terms of Φ_- but only the third component has a simple representation in terms of Φ_-; namely

$$j_\mu^3(x) =: \overline{\psi}_a(x)\gamma_\mu(\frac{\sigma^3}{2})_{ab}\psi_b(x) :=$$

$$= \frac{1}{2} : \overline{\psi}_1(x)\gamma_\mu\psi_1(x)-\overline{\psi}_2(x)\gamma_\mu\psi_2(x) := (2\pi)^{\frac{1}{2}}\epsilon^{\mu\nu}\partial_\nu\Phi_- \quad . \tag{3.170}$$

The other two isospin currents $j_\mu^1(x)$ and $j_\mu^2(x)$ are nonlinear and nonlocal functions of Φ_- [47]; a more symmetrical treatment of the bosonized form of the isotriplet currents is available within the framework of non abelian bosonization [48]. For the multiflavor Schwinger model this approach has been carried out in [49], providing results in agreement with [47].

The excitations are most conveniently classified in terms of the quantum numbers of P-parity and G-parity; G-parity is related to the charge conjugation C by

$$G=e^{i\pi\frac{\sigma^2}{2}}C \quad . \tag{3.171}$$

Φ_- is a G-even pseudoscalar, while Φ_+ is a G-odd pseudoscalar

$$\Phi_- \quad : \quad I^{PG}=1^{-+} \tag{3.172}$$

$$\Phi_+ \quad : \quad I^{PG}=0^{--} \quad . \tag{3.173}$$

The massive meson Φ_+ is stable by G conservation since the action (3.169) is invariant under $\Phi_+ \longrightarrow -\Phi_+$.

In the massive $SU(2)$ Schwinger model $-$ when the mass of the fermion m is small compared to e^2 (strong coupling) $-$ Coleman [47] showed that - in addition to the triplet Φ_- ($I^{PG}=1^{-+}$) the low-energy spectrum exhibits a singlet $I^{PG}=0^{++}$ lying on top of the triplet Φ_-. In this limit the gauge theory is mapped to a sine-Gordon model and the low lying excitations are soliton-antisoliton states. When $m\to 0$, these soliton-antisoliton states become massless [50]; in this limit, the analysis of the many body wave functions, carried out in ref.[50], hints to the existence of a whole class of massless states with positive G-parity; P-parity however cannot be determined with the procedure developed in [50]. These are not the only excitations of the model: way up in mass there is the isosinglet $I^{PG}=0^{--}$, (Φ_+), already discussed in ref. [47]. The model exhibits also triplets, whose mass $-$ of order m_S or greater $-$ stays finite [50]; among the triplets there is a G-even state [2].

[2] K. Harada private communication.

The Hamiltonian, gauge constraint and non-vanishing (anti-)commutators of the continuum two-flavor Schwinger model are

$$H = \int dx \qquad \left[\tfrac{e^2}{2}E^2(x) + \textstyle\sum_{a=1}^2 \psi_a^\dagger(x)\alpha\left(i\partial_x + eA(x)\right)\psi_a(x)\right] \tag{3.174}$$

$$\partial_x E(x) \ + \textstyle\sum_{a=1}^2 \psi_a^\dagger(x)\psi_a(x) \sim 0 \tag{3.175}$$

$$[A(x), E(y)] = i\delta(x-y) \ , \left\{\psi_a(x), \psi_b^\dagger(y)\right\} = \delta_{ab}\delta(x-y) \ \ . \tag{3.176}$$

A lattice Hamiltonian, constraint and (anti-) commutators reducing to (3.174,3.175,3.176) in the naive continuum limit are

$$\begin{aligned} H_S = \frac{e^2 a}{2}\sum_{x=1}^N E_x^2 \quad &- \quad \frac{it}{2a}\sum_{x=1}^N\sum_{a=1}^2 \left(\psi_{a,x+1}^\dagger e^{iA_x}\psi_{a,x} - \psi_{a,x}^\dagger e^{-iA_x}\psi_{a,x+1}\right) \\ E_x - E_{x-1} \quad &+ \quad \psi_{1,x}^\dagger\psi_{1,x} + \psi_{2,x}^\dagger\psi_{2,x} - 1 \sim 0 \ , \\ [A_x, E_y] = i\delta_{x,y} \qquad &, \qquad \left\{\psi_{a,x}, \psi_{b,y}^\dagger\right\} = \delta_{ab}\delta_{xy} \ \ . \end{aligned} \tag{3.177}$$

The fermion fields are defined on the sites, $x = 1, ..., N$, the gauge and electric fields, A_x and E_x, on the links $[x; x+1]$, N is an even integer and, when N is finite it is convenient to impose periodic boundary conditions. When N is finite, the continuum limit is the two-flavor Schwinger model on a circle [44]. The coefficient t of the hopping term in (3.177) plays the role of the lattice light speed. In the naive continuum limit, $e_L = e_c$ and $t = 1$.

The Hamiltonian and gauge constraint exhibit the discrete symmetries

- Parity P:

$$A_x \longrightarrow -A_{-x-1}, \ E_x \longrightarrow -E_{-x-1}, \ \psi_{a,x} \longrightarrow (-1)^x\psi_{a,-x}, \ \psi_{a,x}^\dagger \longrightarrow (-1)^x\psi_{a,-x}^\dagger \tag{3.178}$$

- Discrete axial symmetry Γ:

$$A_x \longrightarrow A_{x+1}, \ E_x \longrightarrow E_{x+1}, \ \psi_{a,x} \longrightarrow \psi_{a,x+1}, \ \psi_{a,x}^\dagger \longrightarrow \psi_{a,x+1}^\dagger \tag{3.179}$$

- Charge conjugation C:

$$A_x \longrightarrow -A_{x+1}, \ E_x \longrightarrow -E_{x+1}, \ \psi_{a,x} \longrightarrow \psi_{a,x+1}^\dagger, \ \psi_{a,x}^\dagger \longrightarrow \psi_{a,x+1} \tag{3.180}$$

- G-parity:

$$\begin{aligned} A_x \longrightarrow -A_{x+1}, \ E_x \longrightarrow -E_{x+1} \\ \psi_{1,x} \longrightarrow \psi_{2,x+1}^\dagger, \ \psi_{1,x}^\dagger \longrightarrow \psi_{2,x+1} \\ \psi_{2,x} \longrightarrow -\psi_{1,x+1}^\dagger, \ \psi_{2,x}^\dagger \longrightarrow -\psi_{1,x+1} \ . \end{aligned} \tag{3.181}$$

The lattice two-flavor Schwinger model is equivalent to a one dimensional quantum Coulomb gas on the lattice with two kinds of particles. To see this one can fix the gauge, $A_x = A$ (Coulomb gauge). Eliminating the non-constant electric field and using the gauge constraint, one obtains the effective Hamiltonian

$$\begin{aligned} H_S \quad &= \quad H_u + H_p \equiv \left[\frac{e_L^2}{2N}E^2 + \frac{e_L^2 a}{2}\sum_{x,y}\rho(x)V(x-y)\rho(y)\right] + \\ &+ \quad \left[-\frac{it}{2a}\sum_x\sum_{a=1}^2(\psi_{a,x+1}^\dagger e^{iA}\psi_{a,x} - \psi_{a,x}^\dagger e^{-iA}\psi_{a,x+1})\right] \ , \end{aligned} \tag{3.182}$$

where the charge density is

$$\rho(x) = \psi_{1,x}^\dagger\psi_{1,x} + \psi_{2,x}^\dagger\psi_{2,x} - 1 \ \ , \tag{3.183}$$

and the potential

$$V(x-y) = \frac{1}{N}\sum_{n=1}^{N-1} e^{i2\pi n(x-y)/N}\frac{1}{4\sin^2\frac{\pi n}{N}} \tag{3.184}$$

is the Fourier transform of the inverse laplacian on the lattice for non zero momentum. The constant modes of the gauge field decouple in the thermodynamic limit $N \longrightarrow \infty$.

3.2 The strong coupling limit and the antiferromagnetic Heisenberg Hamiltonian

In the thermodynamic limit the Schwinger Hamiltonian (3.182), rescaled by the factor $e_L^2 a/2$, reads

$$H = H_0 + \epsilon H_h \tag{3.185}$$

with

$$H_0 = \sum_{x>y}\left[\frac{(x-y)^2}{N} - (x-y)\right]\rho(x)\rho(y) \quad , \tag{3.186}$$

$$H_h = -i(R-L) \tag{3.187}$$

and $\epsilon = t/e_L^2 a^2$. In Eq.(3.187) the right R and left L hopping operators are defined ($L = R^\dagger$) as

$$R = \sum_{x=1}^{N} R_x = \sum_{x=1}^{N}\sum_{a=1}^{2} R_x^{(a)} = \sum_{x=1}^{N}\sum_{a=1}^{2}\psi^\dagger_{a,x+1}e^{iA}\psi_{a,x} \quad . \tag{3.188}$$

On a periodic chain the commutation relation

$$[R, L] = 0 \tag{3.189}$$

is satisfied.

We shall consider the strong coupling perturbative expansion where the Coulomb Hamiltonian (3.186) is the unperturbed Hamiltonian and the hopping Hamiltonian (3.187) the perturbation. Due to Eq.(3.183) every configuration with one particle per site has zero energy, so that the ground state of the Coulomb Hamiltonian (3.186) is 2^N times degenerate. The degeneracy of the ground state can be removed only at the second perturbative order since the first order is trivially zero.

At the second order the lattice gauge theory is effectively described by the antiferromagnetic Heisenberg Hamiltonian. The vacuum energy – at order ϵ^2 – reads

$$E_0^{(2)} = < H_h^\dagger \frac{\Pi}{E_0^{(0)} - H_0} H_h > \tag{3.190}$$

where the expectation values are defined on the degenerate subspace and Π is the operator projecting on a set orthogonal to the states with one particle per site. Due to the vanishing of the charge density on the ground states of H_0, the commutator

$$[H_0, H_h] = H_h \tag{3.191}$$

holds on any linear combination of the degenerate ground states. Consequently, from Eq.(3.190) one finds

$$E_0^{(2)} = -2 < RL > \quad . \tag{3.192}$$

On the ground state the combination RL can be written in terms of the Heisenberg Hamiltonian. By introducing the Schwinger spin operators

$$\vec{S}_x = \psi^\dagger_{a,x}\frac{\vec{\sigma}_{ab}}{2}\psi_{b,x} \tag{3.193}$$

the Heisenberg Hamiltonian H_J reads

$$H_J = \sum_{x=1}^{N}\left(\vec{S}_x\cdot\vec{S}_{x+1}-\frac{1}{4}\right) = = \sum_{x=1}^{N}\left(-\frac{1}{2}L_xR_x-\frac{1}{4}\rho(x)\rho(x+1)\right) \tag{3.194}$$

and, on the degenerate subspace, one has

$$< H_J >= \left\langle \sum_{x=1}^{N}\left(\vec{S}_x\cdot\vec{S}_{x+1}-\frac{1}{4}\right)\right\rangle = \left\langle \sum_{x=1}^{N}\left(-\frac{1}{2}L_xR_x\right)\right\rangle \quad . \tag{3.195}$$

Taking into account that products of L_x and R_y at different points have vanishing expectation values on the ground states, and using Eq.(3.195), Eq.(3.192) reads

$$E_0^{(2)} = 4 < H_J > \quad . \tag{3.196}$$

The ground state of H_J singles out the correct vacuum, on which to perform the perturbative expansion. In one dimension H_J is exactly diagonalizable [17, 20]. In the spin model a flavor 1 particle on a site can be represented by a spin up, a flavor 2 particle by a spin down. The spectrum of H_J exhibits 2^N eigenstates; among these, the spin singlet with lowest energy is the non degenerate ground state $|g.s.>$.

We shall construct the strong coupling perturbation theory of the two-flavor Schwinger model using $|g.s.>$ as the unperturbed ground state. $|g.s.>$ is invariant under translations by one lattice site, which amounts to invariance under discrete chiral transformations. As a consequence, at variance with the one-flavor model [14], chiral symmetry cannot be spontaneously broken even in the infinite coupling limit.

$|g.s.>$ has zero charge density on each site and zero electric flux on each link

$$\rho(x)|g.s.>= 0 \quad , \quad E_x|g.s.>= 0 \qquad (x = 1, ..., N) \quad . \tag{3.197}$$

$|g.s.>$ is a linear combination of all the possible states with $\frac{N}{2}$ spins up and $\frac{N}{2}$ spins down. The coefficients are not explicitly known for general N. In section 2.2 we exhibited $|g.s.>$ explicitly for finite size systems of 4, 6 and 8 sites. The Heisenberg energy of $|g.s.>$ is known exactly and, in the thermodynamic limit, is [23, 18]

$$H_J|g.s.>= (-N \ \ln\ 2)|g.s.> \quad . \tag{3.198}$$

Eq.(3.198) provides the second order correction Eq.(3.196) to the vacuum energy, $E_{g.s.}^{(2)} = -4N\ln 2$.

There exist two kinds of excitations created from $|g.s.>$; one kind involves only spin flipping and has lower energy since no electric flux is created, the other involves fermion transport besides spin flipping and thus has a higher energy. For the latter excitations the energy is proportional to the coupling times the length of the electric flux: the lowest energy is achieved when the fermion is transported by one lattice spacing. Of course only the excitations of the first kind can be mapped into states of the Heisenberg model.

In [18] the antiferromagnetic Heisenberg model excitations have been classified. There it was shown that any excitation may be regarded as the scattering state of quasiparticles of spin-1/2: every physical state contains an even number of quasiparticles and the spectrum exhibits only integer spin states. The two simplest excitations of lowest energy in the thermodynamic limit are a triplet and a singlet [18]; they have a dispersion relation depending on the momenta of the two quasiparticles. For vanishing total momentum (relative to the ground state momentum $P_{g.s.} = 0$ for $\frac{N}{2}$ even, $P_{g.s.} = \pi$ for $\frac{N}{2}$ odd) in the thermodynamic limit they are degenerate with the ground state.

In section 2.2 we showed that even for finite size systems, the excited states can be grouped in families corresponding to the classification given in [18]. We explicitly exhibited all the energy

eigenstates for $N = 4$ and $N = 6$. The lowest lying are a triplet and a singlet, respectively; they have a well defined relative (to the ground state) P-parity and G-parity - 1^{-+} for the triplet and 0^{++} for the singlet. Since they share the same quantum numbers these states can be identified, in the limit of vanishing fermion mass, with the soliton-antisoliton excitations found by Coleman in his analysis of the two-flavor Schwinger model. A related analysis about the parity of the lowest lying states in finite size Heisenberg chains, has been given in [51].

Moreover in [18] a whole class $- \mathcal{M}_{AF} -$ of gapless excitations at zero momentum was singled out in the thermodynamic limit; these states are eigenstates of the total momentum and consequently have positive G-parity at zero momentum. The low lying states of the Schwinger model also contain [50] many massless excitations with positive G-parity; they are identified [19, 15] with the excitations belonging to $\mathcal{M}_{AF}$. The mass of these states in the Schwinger model can be obtained from the differences between the excitation energies at zero momentum and the ground state energy. The energies of the states $|ex.>$ belonging to the class $\mathcal{M}_{AF}$ have the same perturbative expansion of the ground state. Consequently, the states $|ex.>$ at zero momentum up to the second order in the strong coupling expansion have the same energy of the ground state (3.190), $E_{ex}^{(2)} = -4N \ln 2$. To this order the mass gap is zero. Higher order corrections may give a mass gap.

3.3 The meson masses

In this section we determine the masses for the states obtained by fermion transport of one site on the Heisenberg model ground state. Our analysis shows that besides the G-odd pseudoscalar isosinglet 0^{--} with mass $m_S = e_L\sqrt{2/\pi}$, there are also a G-even scalar isosinglet 0^{++} and a pseudoscalar isotriplet 1^{-+} and a G-odd scalar isotriplet 1^{+-} with masses of the order of m_S or greater. The quantum numbers are relative to those of the ground state $I^{PG}_{g.s.} = 0^{++}$ for $N/2$ even $I^{PG}_{g.s.} = 0^{--}$ for $N/2$ odd.

Two states can be created using the spatial component of the vector $j^1(x)$ Eq.(3.159) and isovector $j^1_\alpha(x)$ Eqs.(3.157,3.158) Schwinger model currents. They are the G-odd pseudoscalar isosinglet $I^{PG} = 0^{--}$ and the G-even pseudoscalar isotriplet $I^{PG} = 1^{-+}$. The lattice operators with the correct quantum numbers creating these states at zero momentum, when acting on $|g.s.>$, read

$$S = R + L = \sum_{x=1}^{N} j^1(x) \tag{3.199}$$

$$T_+ = (T_-)^\dagger = R^{(12)} + L^{(12)} = \sum_{x=1}^{N} j^1_+(x) \tag{3.200}$$

$$T_0 = \frac{1}{\sqrt{2}}(R^{(11)} + L^{(11)} - R^{(22)} - L^{(22)}) = \sum_{x=1}^{N} j^1_3(x) \quad . \tag{3.201}$$

$R^{(ab)}$ and $L^{(ab)}$ in (3.200,3.201) are the right and left flavor changing hopping operators ($L^{(ab)} = (R^{(ab)})^\dagger$)

$$R^{(ab)} = \sum_{x=1}^{N} \psi^\dagger_{a,x+1} e^{iA} \psi_{b,x} \quad .$$

The states are given by

$$|S> = |0^{--}>= S|g.s.> \tag{3.202}$$

$$|T_\pm> = |1^{-+}, \pm 1>= T_\pm|g.s.> \tag{3.203}$$

$$|T_0> = |1^{-+}, 0>= T_0|g.s.> \quad . \tag{3.204}$$

They are normalized as

$$<S|S> = <g.s.|S^\dagger S|g.s.>= -4 <g.s.|H_J|g.s.>= 4N \ln 2 \tag{3.205}$$

$$<T_+|T_+> = \frac{2}{3}(N+ <g.s.|H_J|g.s.>) = \frac{2}{3}N(1 - \ln 2) \tag{3.206}$$

and

$$< T_0|T_0 >=< T_-|T_- >=< T_+|T_+ > \quad . \tag{3.207}$$

In Eqs.(3.205,3.206,3.207) $< g.s.|g.s. >= 1$.

The isosinglet energy, up to the second order in the strong coupling expansion, is $E_S = E_S^{(0)} + \epsilon^2 E_S^{(2)}$ with

$$E_S^{(0)} = \frac{< S|H_0|S >}{< S|S >} = 1 \quad , \tag{3.208}$$

$$E_S^{(2)} = \frac{< S|H_h^\dagger \Lambda_S H_h|S >}{< S|S >} \quad , \tag{3.209}$$

$\Lambda_S = \frac{\Pi_S}{E_S^{(0)} - H_0}$ and $1 - \Pi_S$ a projection operator onto $|S >$. On $|g.s. >$

$$[H_0, (\Pi_S H_h)^n S] = (n+1)(\Pi_S H_h)^n, \quad (n = 0, 1, \ldots), \tag{3.210}$$

holds; Eq.(3.209) may then be written in terms of spin correlators as

$$E_S^{(2)} = E_{g.s.}^{(2)} + 4 - \frac{\sum_{x=1}^N < g.s.|\vec{S}_x \cdot \vec{S}_{x+2} - \frac{1}{4}|g.s. >}{< g.s.|H_J|g.s. >} \quad . \tag{3.211}$$

One immediately recognizes that the excitation spectrum is determined once $< g.s.|\vec{S}_x \cdot \vec{S}_{x+2}|g.s. >$ is known. Equations similar to Eq.(3.211) may be established also at a generic order of the strong coupling expansion.

At the zeroth perturbative order the pseudoscalar triplet is degenerate with the isosinglet $E_T^{(0)} = E_S^{(0)} = 1$. Following the same procedure as before one may compute the energy of the states (3.203) and (3.204) to the second order in the strong coupling expansion. To this order, the energy is given by

$$\begin{aligned} E_T^{(2)} &= E_{g.s.}^{(2)} - \Delta_{DS}(T) - \\ &\frac{4 < g.s.|H_J|g.s. > +5 \sum_{x=1}^N < g.s.|\vec{S}_x \cdot \vec{S}_{x+2} - \frac{1}{4}|g.s. >}{N+ < g.s.|H_J|g.s. >} \end{aligned} \tag{3.212}$$

where in terms of the vector operator $\vec{V} = \sum_{x=1}^N \vec{S}_x \wedge \vec{S}_{x+1}$, one can write $\Delta_{DS}(T)$ as

$$\Delta_{DS}(T_\pm) = 12 \frac{< g.s.|(V_1)^2|g.s. > + < g.s.|(V_2)^2|g.s. >}{N+ < g.s.|H_J|g.s. >} \tag{3.213}$$

$$\Delta_{DS}(T_0) = 12 \frac{2 < g.s.|(V_3)^2|g.s. >}{N+ < g.s.|H_J|g.s. >} \quad . \tag{3.214}$$

The VEV of each squared component of $\vec{V}$ on the rotationally invariant singlet $|g.s. >$ give the same contribution *i.e.* $\Delta_{DS}(T_\pm) = \Delta_{DS}(T_0)$: the triplet states (as in the continuum theory) have a degenerate mass gap. This is easily verified by direct computation on finite size systems; when the size of the system is finite one may also show that Δ_{DS} is of zeroth order in N.

The excitation masses are given by $m_S = \frac{e_L^2 a}{2}(E_S - E_{g.s.})$ and $m_T = \frac{e_L^2 a}{2}(E_T - E_{g.s.})$. Consequently, the ($N$-dependent) ground state energy terms appearing in $E_S^{(2)}$ and $E_T^{(2)}$ cancel and what is left are only N independent terms. This is a good check of our computation, being the mass an intensive quantity.

In principle one should expect also excitations created acting on $|g.s. >$ with the chiral currents, in analogy with the one flavor Schwinger model where, as shown in ref. [37], the chiral current creates a two-meson bound state. The chiral currents operators for the two flavor Schwinger model are given by

$$j^5(x) = \overline{\psi}(x)\gamma^5\psi(x) \tag{3.215}$$

$$j^5_\alpha(x) = \overline{\psi}_a(x)\gamma^5(\frac{\sigma}{2})_{ab}\psi_b(x) \quad . \tag{3.216}$$

The corresponding lattice operators at zero momentum are

$$
\begin{aligned}
S^5 &= R - L = \sum_{x=1}^N j^5(x) && (3.217)\\
T^5_+ &= (T^5_-)^\dagger = R^{(12)} - L^{(12)} = \sum_{x=1}^N j^5_+(x) && (3.218)\\
T^5_0 &= \frac{1}{\sqrt{2}}(R^{(11)} - L^{(11)} - R^{(22)} + L^{(22)}) = \sum_{x=1}^N j^5_3(x) \quad . && (3.219)
\end{aligned}
$$

The states created by (3.217,3.218,3.219) when acting on $|g.s.>$, are

$$
\begin{aligned}
|S^5> &= |0^{++}>= S^5|g.s.> && (3.220)\\
|T^5_\pm> &= |1^{+-}, \pm 1>= T^5_\pm|g.s.> && (3.221)\\
|T^5_0> &= |1^{+-}, 0>= T^5_0|g.s.> \quad . && (3.222)
\end{aligned}
$$

They are normalized as

$$
\begin{aligned}
< S^5|S^5> &= < g.s.|S^{5\dagger} S^5|g.s.>= -4 < g.s.|H_J|g.s.>= 4N\log 2 && (3.223)\\
< T^5_+|T^5_+> &= \frac{2}{3}(N+ < g.s.|H_J|g.s.>) = \frac{2}{3}N(1-\log 2) && (3.224)
\end{aligned}
$$

and

$$
< T^5_0|T^5_0>=< T^5_-|T^5_->=< T^5_+|T^5_+> \quad . \tag{3.225}
$$

Following the computational scheme used to study $|S>$ and $|T>$, one finds for the state $|S^5>$

$$
\begin{aligned}
E^{(0)}_{S^5} &= 1 && (3.226)\\
E^{(2)}_{S^5} &= E^{(2)}_{g.s.} + 12 - 3\frac{\sum_{x=1}^N < g.s.|\vec{S}_x\cdot\vec{S}_{x+2} - \frac{1}{4}|g.s.>}{< g.s.|H_J|g.s.>} \quad . && (3.227)
\end{aligned}
$$

For the triplet $|T^5>$ one gets

$$
\begin{aligned}
E^{(0)}_{T^5} &= 1 && (3.228)\\
E^{(2)}_{T^5} &= E^{(2)}_{g.s.} + \frac{\sum_{x=1}^N < g.s.|\vec{S}_x\cdot\vec{S}_{x+2} - \frac{1}{4}|g.s.> -4 < g.s.|H_J|g.s.>}{N+ < g.s.|H_J|g.s.>} \quad . && (3.229)
\end{aligned}
$$

Now we can compute the mass spectrum up to the second order in the strong coupling expansion. Using Eq.(2.102), the isosinglet mass reads as

$$
\frac{m_S}{e^2 a} = \frac{1}{2} + 1.9509 \quad \epsilon^2 \quad . \tag{3.230}
$$

For what concerns the isotriplet mass, since the double sum in Eq.(3.212) is given by

$$
\Delta_{DS}(T) = 8\frac{< g.s.|\vec{V}\cdot\vec{V}|g.s.>}{N+ < g.s.|H_J|g.s.>} \quad , \tag{3.231}
$$

using Eq.(2.123), one gets

$$
\frac{m_T}{e_L^2 a} = \frac{1}{2} + 0.0972 \quad \epsilon^2 \quad . \tag{3.232}
$$

The existence of massive isotriplets was already noticed in [50], and their mass in the continuum theory was numerically computed for various values of the fermion mass. In particular there is a G-parity even isotriplet with mass approximately equal to the mass of the isosinglet 0^{--}.

The masses of the $|S_5>$ isosinglet and the $|T_5>$ isotriplet are

$$
\begin{aligned}
\frac{m_{S^5}}{e^2 a} &= \frac{1}{2} + 5.85\epsilon^2 && (3.233)\\
\frac{m_{T^5}}{e^2 a} &= \frac{1}{2} + 4.4069\epsilon^2 \, . && (3.234)
\end{aligned}
$$

Equations (3.230), (3.232), (3.233) and (3.234) provide the values of m_S, m_T, m_{S^5} and m_{T^5} for small values of $z = \epsilon^2 = \frac{t^2}{e_L^4 a^4}$ up to the second order in the strong coupling expansion. Whereas (3.232) is only approximate (3.230), (3.233) and (3.234) are exact at the second order in the ϵ expansion. We extrapolated these masses to the continuum limit using the standard technique of the Padé approximants [15] and we got results in good agreement with the continuum.

4 The multiflavor lattice Schwinger models

We now study the $\mathcal{N}$-flavor lattice Schwinger models in the hamiltonian formalism using staggered fermions. The $\mathcal{N}$-flavor Schwinger models have many features in in common with four dimensional QCD: at the classical level they have a symmetry group $U_L(\mathcal{N}) \otimes U_R(\mathcal{N}) = SU_L(\mathcal{N}) \otimes SU_R(\mathcal{N}) \otimes U_V(1) \otimes U_A(1)$ that is broken down to $SU_L(\mathcal{N}) \otimes SU_R(\mathcal{N}) \otimes U_V(1)$ by the axial anomaly exactly like in QCD [52]. The massless $\mathcal{N}$-flavor Schwinger models describe no real interactions between their particles as one can infer by writing the model action in a bosonized form. The model exhibits one massive and $\mathcal{N}^2 - 1$ massless pseudoscalar "mesons" [53].

On the lattice we shall prove that $-$ at the second order in the strong coupling expansion $-$ the lattice Schwinger models are effectively described by $SU(\mathcal{N})$ quantum antiferromagnetic spin-1/2 Heisenberg Hamiltonians with spins in a particular fundamental representation of the $SU(\mathcal{N})$ Lie algebra and with nearest neighbours couplings. The features of the model are very different depending on if $\mathcal{N}$ is odd or even. When $\mathcal{N}$ is odd, the ground state energy in the strong coupling limit is proportional to e_L^2, the square of the electromagnetic coupling constant. In contrast, when $\mathcal{N}$ is even the ground state energy in the strong coupling limit is of order 1. This difference arises from the proper definition of the charge density

$$\rho(x) = \sum_{a=1}^{\mathcal{N}} \psi^{\dagger}_{a,x} \psi_{a,x} - \frac{\mathcal{N}}{2} \tag{4.235}$$

where the constant $\mathcal{N}/2$ has been subtracted from the charge density operator in order to make it odd under the charge conjugation transformation. As a consequence, when $\mathcal{N}$ is even, $\rho(x)$ admits zero eigenvalues and the ground state does not support any electric flux, while when $\mathcal{N}$ is odd the ground state exhibits a staggered configuration of the charge density and electromagnetic fluxes.

In the continuum the Coleman theorem [22] prevents the formation of either an isoscalar chiral condensate $< \overline{\psi}\psi >$ or an isovector chiral condensate $< \overline{\psi} T^a \psi > -$ where T^a is an $SU(\mathcal{N})$ generator $-$ for every model with an internal $SU(\mathcal{N})$-flavor symmetry. This feature is reproduced on the lattice also for this class of models [16].

4.1 The continuum $\mathcal{N}$-flavor Schwinger models

The continuum $SU(\mathcal{N})$-flavor Schwinger models are defined by the action

$$S = \int d^2x (\sum_{a=1}^{\mathcal{N}} \overline{\psi}_a (i\gamma_\mu \partial^\mu + \gamma_\mu A^\mu)\psi_a - \frac{1}{4e_c^2} F_{\mu\nu} F^{\mu\nu}) \tag{4.236}$$

where the $\mathcal{N}$ fermions have been introduced in a completely symmetric way. Although the theory described by (4.236) strictly parallels what has been shown in the previous section for the $SU(2)$ model, we shall now report a detailed analysis both for the sake of clarity and to show that some difference appears between $\mathcal{N}$ even and odd.

The Dirac fields are an $\mathcal{N}$-plet, *i.e.* transform according to the fundamental representation of the flavor group while the electromagnetic field is an $SU(\mathcal{N})$ singlet. The flavor symmetry of the theory cannot be spontaneously broken for the same reasons as in the $SU(2)$ case. The particles of the theory belong to $SU(\mathcal{N})$ multiplets. The action is invariant under the symmetry

$$SU_L(\mathcal{N}) \otimes SU_R(\mathcal{N}) \otimes U_V(1) \otimes U_A(1) \tag{4.237}$$

The symmetry generators act as follows

$$
\begin{aligned}
SU_L(\mathcal{N}) \quad &: \quad \psi_a(x) \longrightarrow (e^{i\theta_\alpha T^\alpha P_L})_{ab}\ \psi_b(x)\ ,\ \overline{\psi_a}(x) \longrightarrow \overline{\psi_b}\ (x)(e^{-i\theta_\alpha T^\alpha P_R})_{ba} && (4.238)\\
SU_R(\mathcal{N}) \quad &: \quad \psi_a(x) \longrightarrow (e^{i\theta_\alpha T^\alpha P_R})_{ab}\ \psi_b(x)\ ,\ \overline{\psi_a}(x) \longrightarrow \overline{\psi_b}(x)\ (e^{-i\theta_\alpha T^\alpha P_L})_{ba} && (4.239)\\
U_V(1) \quad &: \quad \psi_a(x)\ \longrightarrow (e^{i\theta(x)\mathbf{1}})_{ab}\ \psi_b(x)\ ,\ \psi_a^\dagger(x) \longrightarrow \psi_b^\dagger(x)\ (e^{-i\theta(x)\mathbf{1}})_{ba} && (4.240)\\
U_A(1) \quad &: \quad \psi_a(x) \longrightarrow (e^{i\alpha\gamma_5\mathbf{1}})_{ab}\ \psi_b(x)\ ,\ \psi_a^\dagger(x) \longrightarrow \psi_b^\dagger(x)\ (e^{-i\alpha\gamma_5\mathbf{1}})_{ba} && (4.241)
\end{aligned}
$$

where T^α are the generators of the $SU(\mathcal{N})$ group, θ_α, $\theta(x)$ and α are real coefficients and

$$P_L = \frac{1}{2}(1-\gamma_5)\ ,\ P_R = \frac{1}{2}(1+\gamma_5) \quad . \tag{4.242}$$

At the classical level the above symmetries lead to conservation laws for the isovector, vector and axial currents

$$
\begin{aligned}
j^\mu_\alpha(x)_R &= \overline{\psi}_a(x)\gamma^\mu P_R(T_\alpha)_{ab}\psi_b(x) \quad , && (4.243)\\
j^\mu_\alpha(x)_L &= \overline{\psi}_a(x)\gamma^\mu P_L(T_\alpha)_{ab}\psi_b(x) \quad , && (4.244)\\
j^\mu(x) &= \overline{\psi}_a(x)\gamma^\mu \mathbf{1}_{ab}\psi_b(x) \quad , && (4.245)\\
j^\mu_5(x) &= \overline{\psi}_a(x)\gamma^\mu\gamma^5 \mathbf{1}_{ab}\psi_b(x) \quad . && (4.246)
\end{aligned}
$$

At the quantum level the vector and axial currents cannot be simultaneously conserved. If the regularization is gauge invariant, so that the vector current is conserved, then the axial current acquires the anomaly which breaks the symmetry $U_A(1)$ [52]

$$\partial_\mu j^\mu_5(x) = \mathcal{N}\frac{e_c^2}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}(x) \quad . \tag{4.247}$$

The isoscalar $<\overline{\psi}\psi>$ and isovector $<\overline{\psi}T^\alpha\psi>$ chiral condensates are zero due to the Coleman theorem [22], in fact they would break not only the $U_A(1)$ symmetry of the action but also the continuum internal symmetry $SU_L(\mathcal{N})\otimes SU_R(\mathcal{N})$ down to $SU_V(\mathcal{N})$. There is an order parameter just for the breaking of the $U_A(1)$ symmetry [42, 46], the operator

$$<\overline{\psi}_L^{(\mathcal{N})}\ldots\overline{\psi}_L^{(1)}\psi_R^{(1)}\ldots\psi_R^{(\mathcal{N})}>=(\frac{e^\gamma}{4\pi})^{\mathcal{N}}(\sqrt{\frac{\mathcal{N}}{\pi}}e_c)^{\mathcal{N}} \quad . \tag{4.248}$$

Under a discrete chiral rotation

$$\psi_L \to \gamma_5\psi_L = -\psi_L \quad , \quad \psi_R \to \gamma_5\psi_R = \psi_R \tag{4.249}$$

the operator (4.248) of course transforms as

$$<\overline{\psi}_L^{(\mathcal{N})}\ldots\overline{\psi}_L^{(1)}\psi_R^{(1)}\ldots\psi_R^{(\mathcal{N})}>\to(-1)^{\mathcal{N}}<\overline{\psi}_L^{(\mathcal{N})}\ldots\overline{\psi}_L^{(1)}\psi_R^{(1)}\ldots\psi_R^{(\mathcal{N})}> \tag{4.250}$$

The operator (4.248) is even under (4.249) when $\mathcal{N}$ is even and this implies that notwithstanding the fact that the continuous chiral rotations $U_A(1)$ are broken by the non-zero VEV (4.248), the discrete chiral symmetry (4.249) is unbroken. When $\mathcal{N}$ is odd also the discrete chiral symmetry (4.249) is broken by the non-zero VEV (4.248).

The usual abelian bosonization procedure may again be applied provided that $\mathcal{N}$ Bose fields are introduced [52, 53, 54]

$$:\overline{\psi}_a\gamma^\mu\psi_a := \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\Phi_a\ ,\ a=1,\ldots\mathcal{N} \tag{4.251}$$

The electric charge density and the action read

$$j_0 =: \sum_{a=1}^{\mathcal{N}}\psi_a^\dagger\psi_a := \frac{1}{\sqrt{\pi}}\partial_x(\sum_{a=1}^{\mathcal{N}}\Phi_a) \quad , \tag{4.252}$$

$$S = \int d^2x(\frac{1}{2}\sum_{a=1}^{\mathcal{N}} \partial_\mu \Phi_a \partial^\mu \Phi_a + \frac{e_c^2}{2\pi}(\sum_{a=1}^{\mathcal{N}} \Phi_a)^2) \tag{4.253}$$

The mass matrix is determined by the last term in Eq.(4.253) and must be diagonalized. The field degrees of freedom span the vector space on which the mass matrix is defined. The action must be expressed in terms of an orthonormal basis of field vectors, in order to have a properly normalised kinetic energy term. The original Φ^a in Eq.(4.253) are orthonormal basis vectors, but they are not eigenvectors of the mass matrix. The mass matrix has one non-zero eigenvalue $\frac{e_c^2}{\mathcal{N}\pi}$ with associated eigenvector $\frac{1}{\sqrt{\mathcal{N}}}\sum_{a=1}^{\mathcal{N}} \Phi^a$ and all the other eigenvalues are zero. The remaining eigenvectors can be made orthonormal by the following change of variables

$$\tilde{\Phi}^a = O^a_b \Phi^b \tag{4.254}$$

where the orthogonal matrices O^a_b are [54]

$$\begin{aligned}
O^1_b &= \frac{1}{\sqrt{\mathcal{N}}}(1,1,\ldots,1) \quad , & (4.255)\\
O^2_b &= \frac{1}{\sqrt{\mathcal{N}(\mathcal{N}-1)}}(1,1,\ldots,-\mathcal{N}+1) \quad , & (4.256)\\
O^3_b &= \frac{1}{\sqrt{(\mathcal{N}-1)(\mathcal{N}-2)}}(1,1,\ldots,-\mathcal{N}+2,0) \quad , & (4.257)\\
&\vdots \\
O^{\mathcal{N}}_b &= \frac{1}{\sqrt{2}}(1,-1,0,\ldots,0) \quad . & (4.258)
\end{aligned}$$

In terms of these new fields $\tilde{\Phi}^a$ the action (4.253) reads

$$S = \int d^2x(\frac{1}{2}\sum_{a=1}^{\mathcal{N}} \partial_\mu \tilde{\Phi}_a \partial^\mu \tilde{\Phi}^a - \frac{1}{2}\mu^2(\tilde{\Phi}^1)^2) \tag{4.259}$$

where $\mu^2 = \mathcal{N}\frac{e_c^2}{\pi}$. The action (4.259) describes $\mathcal{N}$ non interacting fields, one massive and $\mathcal{N}-1$ massless. The multiflavor Schwinger model can also be studied in the framework of non abelian bosonization [48], where the relationship between isovector currents and bosonic excitations appears in a more symmetrical form [49].

The Hamiltonian, gauge constraint and non-vanishing (anti-)commutators of the continuum $\mathcal{N}$-flavor Schwinger models are

$$H = \int dx[\frac{e^2}{2}E^2(x) + \sum_{a=1}^{\mathcal{N}} \quad \psi_a^\dagger(x)\alpha\,(i\partial_x + eA(x))\,\psi_a(x)] \tag{4.260}$$

$$\partial_x E(x) + \quad \sum_{a=1}^{\mathcal{N}} \psi_a^\dagger(x)\psi_a(x) \sim 0 \tag{4.261}$$

$$[A(x),E(y)] = i\delta(x-y)\ , \quad \left\{\psi_a(x),\psi_b^\dagger(y)\right\} = \delta_{ab}\delta(x-y) \tag{4.262}$$

4.2 The lattice $\mathcal{N}$-flavor Schwinger models

On the lattice the Hamiltonian, constraint and (anti-) commutators reducing to (4.260,4.261,4.262) in the naive continuum limit are

$$\begin{aligned}
H_S = \frac{e_L^2 a}{2}\sum_{x=1}^{N} E_x^2 \quad &- \quad \frac{it}{2a}\sum_{x=1}^{N}\sum_{a=1}^{\mathcal{N}}(\psi^\dagger_{a,x+1}e^{iA_x}\psi_{a,x} - \psi^\dagger_{a,x}e^{-iA_x}\psi_{a,x+1})\\
E_x - E_{x-1} \quad &+ \quad \sum_{a=1}^{\mathcal{N}}\psi^\dagger_{a,x}\psi_{a,x} - \frac{\mathcal{N}}{2} \sim 0\ , \qquad (4.263)\\
[A_x, E_y] = i\delta_{x,y} \quad &, \quad \left\{\psi_{a,x},\psi^\dagger_{b,y}\right\} = \delta_{ab}\delta_{xy}
\end{aligned}$$

The fermion fields are defined on the sites, $x = 1, \ldots, N$, gauge and the electric fields, A_x and E_x, on the links $[x; x+1]$, N is an even integer and, when N is finite it is convenient to impose periodic boundary conditions. When N is finite, the continuum limit is the $\mathcal{N}$-flavor Schwinger model on a circle [44]. The coefficient t of the hopping term in (4.263) plays the role of the lattice light speed. In the naive continuum limit, $e_L = e_c$ and $t = 1$.

The lattice $\mathcal{N}$-flavor Schwinger model is equivalent to a one dimensional quantum Coulomb gas on the lattice with $\mathcal{N}$ kinds of particles. To see this one can fix the gauge, $A_x = A$ (Coulomb gauge). Eliminating the non-constant electric field and using the gauge constraint, one obtains the effective Hamiltonian

$$
\begin{aligned}
H_S &= H_u + H_p \equiv \left[\frac{e_L^2}{2N}E^2 + \frac{e_L^2 a}{2}\sum_{x,y}\rho(x)V(x-y)\rho(y)\right] + \\
&+ \left[-\frac{it}{2a}\sum_{x}\sum_{a=1}^{\mathcal{N}}(\psi^\dagger_{a,x+1}e^{iA}\psi_{a,x} - \psi^\dagger_{a,x}e^{-iA}\psi_{a,x+1})\right] \quad , \qquad (4.264)
\end{aligned}
$$

where $\rho(x)$ is given in Eq.(4.235) and the Coulomb potential $V(x-y)$ is given in Eq.(3.184). The constant electric field is normalized so that $[A, E] = i$. The constant modes of the gauge field decouple in the thermodynamic limit $N \to \infty$. In the thermodynamic limit the Schwinger Hamiltonian (4.264), rescaled by the factor $e_L^2 a/2$, reads

$$
H = H_0 + \epsilon H_h \qquad (4.265)
$$

with

$$
\begin{aligned}
H_0 &= \sum_{x>y}[\frac{(x-y)^2}{N} - (x-y)]\rho(x)\rho(y) \quad , && (4.266)\\
H_h &= -i(R-L) && (4.267)
\end{aligned}
$$

and $\epsilon = t/e_L^2 a^2$. In Eq.(4.267) the right R and left L hopping operators are defined ($L = R^\dagger$) as

$$
R = \sum_{x=1}^{N} R_x = \sum_{x=1}^{N}\sum_{a=1}^{\mathcal{N}} R_x^{(a)} = \sum_{x=1}^{N}\sum_{a=1}^{\mathcal{N}} \psi^\dagger_{a,x+1}e^{iA}\psi_{a,x} \quad . \qquad (4.268)
$$

On a periodic chain the commutation relation

$$
[R, L] = 0 \qquad (4.269)
$$

is satisfied.

When $\mathcal{N}$ is even the ground state of the Hamiltonian (4.266) is the state $|g.s.>$ with $\rho(x) = 0$ on every site, *i.e.* with every site half-filled

$$
\sum_{a=1}^{\mathcal{N}} \psi^\dagger_{ax}\psi_{ax}|g.s.> = \frac{\mathcal{N}}{2}|g.s.> \quad . \qquad (4.270)
$$

It is easy to understand that $\rho(x) = 0$ on every site in the ground state by observing that the Coulomb Hamiltonian (4.266) is a non-negative operator and that the states with zero charge density are zero eigenvalues of (4.266). $|g.s.>$ is an highly degenerate state; in fact at each site x the quantum configuration is

$$
\prod_{a=1}^{\frac{\mathcal{N}}{2}} \psi^\dagger_{ax}|0> \quad . \qquad (4.271)
$$

The state (4.271) is antisymmetric in the indices $a = 1, \ldots, \frac{\mathcal{N}}{2}$; *i.e.* it takes on any orientation of the vector in the representation of the flavor symmetry group $SU(\mathcal{N})$ with Young tableau given in fig.(4). The energy of $|g.s.>$ is of order 1, since it is non zero only at the second order in the strong

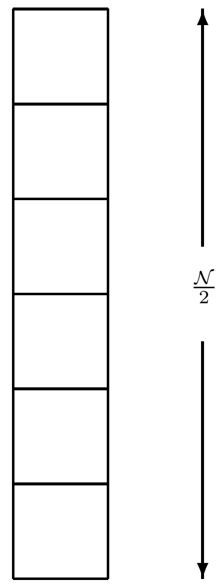

Figure 4: The representation of $SU(\mathcal{N})$ at each site when $\mathcal{N}$ is even

coupling expansion.

When $\mathcal{N}$ is odd the ground states of the Hamiltonian (4.266) are characterized by the staggered charge distribution

$$\rho(x) = \pm\frac{1}{2}(-1)^x \tag{4.272}$$

since (4.272) minimizes the Coulomb Hamiltonian (4.266); one can have $\rho(x) = +1/2$ on the even sublattice and $\rho(x) = -1/2$ on the odd sublattce or viceversa. The electric fields generated by the charge distribution (4.272) are

$$E_x = \pm\frac{1}{4}(-1)^x \tag{4.273}$$

Since now

$$H_0|g.s. >= \frac{1}{16}|g.s. > \tag{4.274}$$

the ground state energy is of order e_L^2. The states $|g.s. >$ are highly degenerate since they can take up any orientation in the vector space which carries the representation of the $SU(\mathcal{N})$ group with the Young tableaux given in fig.(5).

Either when $\mathcal{N}$ is even or when $\mathcal{N}$ is odd the ground state degeneracy is resolved at the second order in the strong coupling expansion. First order perturbations to the vacuum energy vanish. The vacuum energy at order ϵ^2 reads

$$E_0^{(2)} =< H_h^\dagger \frac{\Pi}{E_0^{(0)} - H_0} H_h > \tag{4.275}$$

where the expectation values are defined on the degenerate subspace of ground states and Π is a projection operator projecting orthogonal to the states of the degenerate subspace. Due to the commutation relation

$$[H_0, H_h] = \frac{N-1}{N} H_h - 2\sum_{x,y} \left[V(x-y) - V(x-y-1)\right](L_y + R_y)\rho(x) \tag{4.276}$$

Eq.(4.275) can be rewritten as

$$E_0^{(2)} = -2 < RL > \quad . \tag{4.277}$$

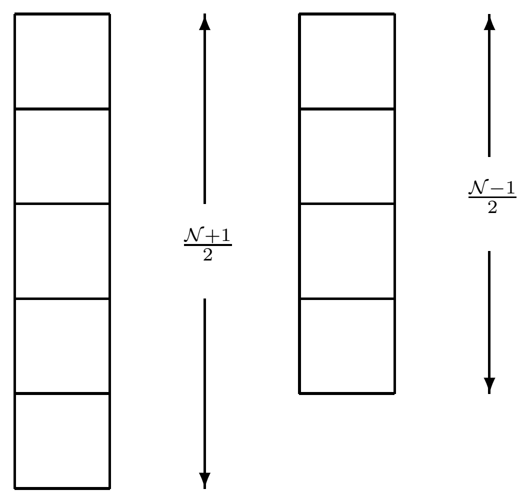

Figure 5: The representation of $SU(\mathcal{N})$ at each site of the even sublattice and odd sublattice when $\mathcal{N}$ is odd

On the ground state the combination RL can be written in terms of the Heisenberg Hamiltonian of a generalized $SU(\mathcal{N})$ antiferromagnet. By introducing as in the previous section the Schwinger spin operators

$$\vec{S}_x = \psi^\dagger_{ax} T^\alpha_{ab} \psi_{bx} \tag{4.278}$$

where T^α are now the generators of the $SU(\mathcal{N})$ group, the $SU(\mathcal{N})$ Heisenberg Hamiltonian reads

$$H_J = \sum_{x=1}^{N} (\ \vec{S}_x \cdot \vec{S}_{x+1} - \frac{\mathcal{N}}{8} + \frac{1}{2\mathcal{N}} \rho(x)\rho(x+1)\ ) = -\frac{1}{2} \sum_{x=1}^{N} L_x R_x \tag{4.279}$$

When $\mathcal{N}$ is even, on the degenerate ground states one has

$$< H_J >=< \sum_{x=1}^{N} (\vec{S}_x \cdot \vec{S}_{x+1} - \frac{\mathcal{N}}{4}) >=< -\frac{1}{2} \sum_{x=1}^{N} L_x R_x > \tag{4.280}$$

while when $\mathcal{N}$ is odd one has

$$< H_J >=< \sum_{x=1}^{N} (\vec{S}_x \cdot \vec{S}_{x+1} - \frac{\mathcal{N}^2+1}{8\mathcal{N}}) >=< -\frac{1}{2} \sum_{x=1}^{N} L_x R_x > \quad . \tag{4.281}$$

Taking into account that the products of L_x and R_y at different points have vanishing expectation values on the ground states and using Eq.(4.280) or Eq.(4.281), Eq.(4.277) reads

$$E_0^{(2)} = 4 < H_J > \quad . \tag{4.282}$$

The problem of determining the correct ground state, on which to perform the perturbative expansion, is then reduced again to the diagonalization of the $SU(\mathcal{N})$ Heisenberg spin-1/2 Hamiltonian (4.279). As we already pointed out in section (2.4), generalized $SU(\mathcal{N})$ antiferromagnetic chains have not been yet analysed in the literature in such a detailed way as the $SU(2)$ chains. Consequentely, the study of the lattice $SU(\mathcal{N})$ flavor lattice Schwinger models become extremely complicated for $\mathcal{N} > 2$. Nonetheless, the computational scheme that we developed for the $U(1)$- and $SU(2)$-flavor models [14, 15, 16], should work for a generic $SU(\mathcal{N})$-flavor model.

The ground state of the gauge models is very different depending on if $\mathcal{N}$ is even or odd. When $\mathcal{N}$ is even, the ground state $|G.S.>$ of the spin Hamiltonian (4.279) is non-degenerate and translationally invariant, and since it is the ground state of the gauge model in the infinite coupling limit,

there is no spontaneous breaking of the chiral symmetry for any $SU(2\mathcal{N})$-flavor lattice Schwinger model. In contrast, when $\mathcal{N}$ is odd, the ground state $|G.S.>$ of the spin Hamiltonian (4.279) is degenerate of order two and is not translationally invariant and consequently any $SU(2\mathcal{N}+1)$-flavor lattice Schwinger model exhibits spontaneous symmetry breaking of the discrete axial symmetry. By translating of one lattice spacing $|G.S.>$ one gets the other one. The $\mathcal{N}$-flavor lattice Schwinger models excitations are also generated from $|G.S.>$ by two very different mechanisms, that we already described for the two-flavor model in section (3.2). There are excitations involving only flavor changes of the fermions without changing the charge density $\rho(x)$ which corrispond to spin flips in the $SU(2)$ invariant model. These excitations are massless. At variance massive excitations involve fermion transport besides flavor changes and are created by applying to $|G.S.>$ the latticized currents of the Schwinger models which vary the on site value of $\rho(x)$.

Very different is the case of the massive multiflavor Schwinger models [55]. When $\mathcal{N}$ is odd, the presence of a non-zero fermionic mass m removes the degeneracy and selects one of the two $|G.S.>$ as the non degenerate ground state. When $\mathcal{N}$ is even the ground state remains translationally invariant in the strong coupling limit $e_L^2 \gg m$. In the weak coupling limit $m \gg e_L^2$ the discrete chiral symmetry is broken for every $\mathcal{N}$.

5 Concluding remarks

In these lectures we analysed the correspondence between the multiflavor strongly coupled lattice Schwinger models and the antiferromagnetic Heisenberg Hamiltonians to investigate the spectrum of the gauge models.

Using the analysis of the excitations of the finite size chains, we showed the equality of the quantum numbers of the states of the Heisenberg model and the low-lying excitations of the two-flavor Schwinger model. We provided also the spectrum of the massive excitations of the gauge model; in order to extract numerical values for the masses, we explicitly computed the pertinent spin-spin correlators of the Heisenberg chain.

The massless and the massive excitations of the gauge model are created from the spin chain ground state with two very different mechanisms: massless excitations involve only spin flipping while massive excitations are created by fermion transport besides spin flipping and do not belong to the spin chain spectrum. As in the continuum theory, due to the Coleman theorem [22], the massless excitations are not Goldstone bosons, but may be regarded as the gapless quantum excitations of the spin-1/2 antiferromagnetic Heisenberg chain [56].

In computing the chiral condensate one can show [15, 16] that, also in the lattice theory, the expectation value of the umklapp operator is different from zero, while both $<\overline{\psi}\psi>$ and $<\overline{\psi}\sigma^a\psi>$ are zero to every order in the strong coupling expansion. This implies that both on the lattice and the continuum the $SU(2)$ flavor symmetry is preserved whereas the $U_A(1)$ axial symmetry is broken. The umklapp operator [15, 16] is the order parameter for this symmetry, but being quadri-linear in the fermi fields, is invariant, in the continuum, under chiral rotation of $\pi/2$ and on the lattice under the corresponding discrete axial symmetry (translation by one lattice site). This shows that the discrete axial symmetry is not broken in both cases. Our lattice computation enhance this result since the ground state of the strongly coupled two-flavor Schwinger model is translationally invariant.

The pattern of symmetry breaking of the continuum is exactly reproduced even if the Coleman theorem does not apply on the lattice and the anomalous symmetry breaking is impossible due to the Nielsen-Ninomiya [57, 58] theorem. At variance with the strongly coupled one-flavor lattice Schwinger model, the anomaly is not realized in the lattice theory via the spontaneous breaking of a residual chiral symmetry [14], but, rather, by explicit breaking of the chiral symmetry due to staggered fermions. The non-vanishing of the VEV of the umklapp operator [15, 16] may be regarded as the only relic, in the strongly coupled lattice theory, of the anomaly of the continuum two-flavor Schwinger model. It is due to the coupling induced by the gauge field, between the right and left-movers on the lattice.

When the fermion mass m is different from zero, some further difference arises between $\mathcal{N}$ odd and $\mathcal{N}$ even. When $\mathcal{N}$ is odd, the mass term induces a translational non invariant ground state, generating a spontaneous chiral symmetry breaking. When $\mathcal{N}$ is even, the ground state remains translationally invariant in the strong coupling limit, *i.e.* $e^2 \gg m$. In the weak coupling limit, $m \gg e^2$, the discrete chiral symmetry is spontaneously broken for every $\mathcal{N}$. For $\mathcal{N} = 2$, the soliton-antisoliton excitations [26] acquire a mass.

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